Solutions to HW3

1.34 First note that if X is empty or contains only one point then the only top on X is $T = \{\phi, X\}$, which is discrete, and the top. sp is Hausdorff (by default). So, its assumed X contains at least two points.

If T is Hausdorff then singletons are closed, so all finite sets are closed. Hence all finite sets are open $\therefore$ T is discrete

If T is discrete then given any two points in X, they are open, so the are open neighbourhoods that don't intersect $\therefore$ T Hausdoff

1.36 Suppose it is Hausdorff. Given any two points, say $x=1$, $y=2$, then $\exists$ open U_1 & $U_2 \ni 1 \in U_1$, $2 \in U_2$ and $U_1 \cap U_2 = \phi$. Now, U_1 open $\Rightarrow$ U_1^c finite and $U_2 \subset U_1^c \Rightarrow U_2$ finite $\Rightarrow$ U_2^c uncountable $\divideontimes$

2.1 a) $\mathring{A} = (0,1)$ and $\bar{A} = [0,1]$

b) $\mathring{A} = A$ and $\bar{A} = X$

c) $\mathring{A} = \{a\}$ and $\bar{A} = X$

g) $\mathring{A} = \phi$ and $\bar{A} = A$

2.4 $p \in A \Rightarrow \mathring{A} = A$ (because A is open so $\mathring{A} = A$) and $\bar{A} = X$

$p \notin A \Rightarrow \mathring{A} = \phi$ (because $\mathring{A}$ and must be ϕ or contain p)
$\bar{A} = A$

2.11 a) $\overline{X - A} = X - \mathring{A}$

$(\subset)$ $x \in \overline{X - A} \Rightarrow \forall$ open U with $x \in U$, $U \cap (X - A) \neq \emptyset$

$\Rightarrow x \notin \mathring{A}$ (otherwise $\exists$ open U with $U \subset A \Rightarrow U \cap (X - A) = \emptyset$)

$\Rightarrow x \in X - \mathring{A}$

$(\supset)$ $x \in X - \mathring{A} \Rightarrow x \notin \mathring{A} \Rightarrow \forall$ open U with $x \in U$, $U \not\subset A$

$\Rightarrow U \cap A^c \neq \emptyset \Rightarrow x \in \overline{X - A}$

b) $\mathrm{Int}(A) \cap \mathrm{Int}(B) = \mathrm{Int}(A \cap B)$

$(\subset)$ $x \in \mathring{A} \cap \mathring{B} \subset A \cap B$ (since $\mathring{A} \subset A$ & $\mathring{B} \subset B$)

since $\mathring{A} \cap \mathring{B}$ is open then $\mathring{A} \cap \mathring{B} \subset (A \cap B)^{\circ} \Rightarrow x \in (A \cap B)^{\circ}$

$(\supset)$ $x \in (A \cap B)^{\circ} \Rightarrow \exists$ open $W \subset A \cap B \ni x \in W$

$\Rightarrow W \subset A$ & $W \subset B$

$\Rightarrow x \in \mathring{A}$ & $x \in \mathring{B}$

$\Rightarrow x \in \mathring{A} \cap \mathring{B}$