Solutions to H/W1

1. $f(A \cap B) \subset f(A) \cap f(B)$

pf: if $A \cap B = \emptyset$ then (by definition) $f(A \cap B) = \emptyset$ and result follows; if $A \cap B \neq \emptyset$ then let $y \in f(A \cap B) \Rightarrow \exists\, x \in A \cap B \ni f(x) = y \Rightarrow$ $x \in A$ & $x \in B \Rightarrow f(x) \in f(A)$ & $f(x) \in f(B)$ $\Rightarrow f(A \cap B) \subset f(A) \cap f(B)$

$f(A) \cap f(B) \subset f(A \cap B)$

pf: $f(A) \cap f(B) = \emptyset \Rightarrow$ result so assume non empty

Now, $y \in f(A) \cap f(B) \Rightarrow y \in f(A)$ & $y \in f(B)$

$\Rightarrow \exists\, x_1 \in A \ni f(x_1) = y$ and $\exists\, x_2 \in B \ni f(x_2) = y$

$\Rightarrow x_1 = x_2$ because f is 1-1

$\Rightarrow \exists\, x \in A \cap B \ni f(x) = y$

$\Rightarrow f(A) \cap f(B) \subset f(A \cap B)$

example: let $f = x^2$, $A = [-1, 0]$ & $B = [0, 1]$

$f(A) = [0,1]$, $f(B) = [0,1]$, $A \cap B = \{0\}$ & $f(A \cap B) = 0$

$\Rightarrow f(A) \cap f(B) \neq f(A \cap B)$

2. If S is countable then $\exists\, f : \mathbb{Z}^+ \to S$ & f is onto

To prove f misses at least one of the sequences $\{s_1, s_2, s_3, \dots\}$ let

$$s_n = \begin{cases} 1 & \text{if } n^{th} \text{ element of } f(n) \text{ equals } 0 \\ 0 & \text{otherwise} \end{cases}$$

By construction f misses this sequence, hence such an f is not possible $\Rightarrow$ S uncountable

1.3 ($\Rightarrow$) T discrete $\Rightarrow$ T contains all subsets of X

$\Rightarrow \{x\} \in T \quad \forall\, x \in X$

($\Leftarrow$) given any nonempty set $A \in X$ then

$$A = \bigcup_{x \in A} \{x\}$$

Since each $\{x\}$ is open, and unions of open sets are open, then A is open $\Rightarrow$ T is discrete top.

1.7 the checklist

i) $\phi \overset{?}{\in} T$: yes, by definition

ii) $X \overset{?}{\in} T$: yes because $p \in X$ so $X \in T$

iii) $U_1, U_2 \in T \overset{?}{\Rightarrow} U_1 \cap U_2 \in T$:

U_1 or U_2 empty $\Rightarrow U_1 \cap U_2 = \phi \Rightarrow U_1 \cap U_2 \in T$

if $U_1 \cap U_2 \neq \phi$ then $p \in U_1$ & $p \in U_2 \Rightarrow$
$p \in U_1 \cap U_2 \Rightarrow U_1 \cap U_2 \in T$

iv) $U_\alpha \in T \overset{?}{\Rightarrow} \cup U_\alpha \in T$:

if $U_\alpha = \phi\ \forall \alpha \Rightarrow \cup U_\alpha = \phi \Rightarrow \cup U_\alpha \in T$

if $\exists \alpha \ni U_\alpha \neq \phi$ then $p \in U_\alpha \Rightarrow p \in \cup U_\alpha$
$\Rightarrow \cup U_\alpha \in T$

1.7 the check list

i) $\phi \overset{?}{\in} T$: yes, by def.

ii) $X \overset{?}{\in} T$: yes, by def

iii) $U_1, U_2 \in T \overset{?}{\Rightarrow} U_1 \cap U_2 \in T$

if U_1, U_2 empty (either one) $\Rightarrow U_1 \cap U_2 = \phi \in T$

if U_1 & U_2 nonempty $\Rightarrow U_i = (-\infty, p_i)$ or $U_i = \mathbb{R}$
$\Rightarrow$

$$U_1 \cap U_2 = \begin{cases} (-\infty, \min\{p_1, p_2\}) & \text{if } U_1 = (-\infty, p_1)\ U_2 = (-\infty, p_2) \\ (-\infty, p_1) & \text{if } U_1 = (-\infty, p_1), U_2 = \mathbb{R} \\ (-\infty, p_2) & \text{if } U_1 = \mathbb{R}, U_2 = (-\infty, p_2) \end{cases}$$

in all cases, $U_1 \cap U_2 \in T$

iv) $U_\alpha \in T \overset{?}{\Rightarrow} \bigcup U_\alpha \in T$

if $U_\alpha = \phi \ \forall \phi \Rightarrow \bigcup U_\alpha = \phi \in T$

if $U_\alpha = \mathbb{R}$ for some $\alpha \Rightarrow \bigcup U_\alpha = \mathbb{R} \in T$

if $U_\alpha = (-\infty, p_\alpha) \ \forall \alpha \Rightarrow \bigcup U_\alpha = (-\infty, p_\infty)$
where $p_\infty = \sup\{p_\alpha\}$

in all cases $\bigcup U_\alpha \in T$