Perturbation Methods
HW 7
Due April 23

1. This excersie considers homogenization applied to the following:

$$\frac{d}{dx}\left(D\frac{du}{dx}\right) + p\frac{du}{dx} + qu = f, \quad \text{for } 0 < x < 1,$$

where

$$u(0) = a \quad \text{and} \quad u(1) = b.$$

In the above, $D = D(x, x/\varepsilon)$, $p = p(x, x/\varepsilon)$, $q = q(x, x/\varepsilon)$ are smooth bounded functions, with D positive.

(a) Assuming $f = f(x, x/\varepsilon)$, determine the resulting homogenized problem.

(b) How does the homogenized problem change if $f = g(x, x/\varepsilon)u^n$?

(c) Explain why homogenization does not work so well in the case of when $f = g(x, x/\varepsilon, u)$.

2. In Section 5.3, it is shown that the homogenized problem has the form

$$\nabla_x \cdot (\mathbf{D}\nabla_x u_0) = f(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Omega,$$

where homogenized coefficients are

$$\mathbf{D} = \begin{pmatrix} \langle D\rangle_p + \langle D\partial_{y_1} a_1\rangle_p & \langle D\partial_{y_1} a_2\rangle_p \\ \langle D\partial_{y_2} a_1\rangle_p & \langle D\rangle_p + \langle D\partial_{y_2} a_2\rangle_p \end{pmatrix}.$$

(a) Suppose D is independent of y_2, that is, $D = D(x_1, x_2, y_1)$. What does $\mathbf{D}$ reduce to in this case?

(b) The Voigt-Riess inequality states that the eigenvalues λ of $\mathbf{D}$ satisfy $\langle D^{-1}\rangle_p^{-1} \le \lambda \le \langle D\rangle_p$ (i.e., the eigenvalues are bounded by the harmonic and arithmetic averages of D). Does your matrix in part (a) satisfy this inequality?

3. 6.6