Solutions to HW6

1. $\varepsilon^2 y'' = q(x,\varepsilon) y \qquad q \sim q_0(x) + \varepsilon q_1(x)$

$y \sim e^{\theta/\varepsilon^\alpha}(y_0 + \varepsilon^\alpha y_1 + \cdots) \Rightarrow \alpha = 1$ and

$O(1)\quad \theta_x^2 = q_0 \Rightarrow \theta_x = \pm\sqrt{q_0} \Rightarrow \theta = \pm\int\sqrt{q_0}$

$O(\varepsilon)\quad \theta'' y_0 + 2\theta' y_0' = q_1 y_0 \Rightarrow y_0' = \left(\frac{q_1}{2\theta'} - \frac{1}{2}\frac{\theta''}{\theta'}\right) y_0 \Rightarrow$

$$y_0 = \frac{\alpha}{\sqrt{\theta_x}} \exp\left(\frac{1}{2}\int\frac{q_1}{\theta'}\right) = \frac{\alpha}{\sqrt{\theta_x}}\exp\left(\pm\frac{1}{2}\int\frac{q_1}{\sqrt{q_0}}\right)$$

so

$$y \sim \frac{1}{q_0^{1/4}}\left[\alpha e^{\int\left(\frac{1}{2}\sqrt{q_0} + \frac{1}{2}\frac{q_1}{\sqrt{q_0}}\right)} + \beta e^{-\int\left(\frac{1}{2}\sqrt{q_0} + \frac{1}{2}\frac{q_1}{\sqrt{q_0}}\right)}\right]$$

Also,
$$K = \int^x \sqrt{q} \sim \int^x \sqrt{q_0 + \varepsilon q_1} \sim \int^x \sqrt{q_0} + \frac{1}{2}\frac{\varepsilon q_1}{\sqrt{q_0}} + \cdots$$
$$= K + O(\varepsilon^2)$$

means it can replace ~~K~~ K

2. a) $y = hw \Rightarrow y' = h'w + hw'$ & $y'' = h''w + 2h'w' + hw'' \Rightarrow$

$$phw' + \underbrace{(2ph' + hp')}_{=0\ \Rightarrow\ h = h_0/\sqrt{p}}w' + (\underbrace{ph'' + p'h'}_{\frac{d}{dx}(ph')} - rh)w = -\lambda^2 qhw$$

b) Assuming $w \sim e^{\lambda^\alpha\theta}(w_0 + \lambda^{-\alpha}w_1 + \cdots)$ (so $\lambda \Leftrightarrow \frac{1}{\varepsilon}$)

$O(\lambda^2)\quad p\theta_x^2 + q = 0 \Rightarrow \theta_x^2 = -\frac{q}{p} \Rightarrow \theta_x = \pm i\sqrt{\frac{q}{p}}$

$O(\lambda)\quad p(\theta'' w_0 + 2\theta' w_0') = 0 \Rightarrow w_0' = -\frac{\theta''}{2\theta'}w_0 \Rightarrow$

$$w_0 = \bar{\alpha}\cdot\frac{1}{\sqrt{\theta_x}} = \bar{\bar{\alpha}}\cdot\left(\frac{p}{q}\right)^{1/4}$$

$\Rightarrow$
$$w \sim \left(\frac{p}{q}\right)^{1/4}\left[\alpha\cos\left(\lambda\int_0^x\sqrt{\frac{q}{p}}\,dx\right) + \beta\sin\left(\lambda\int_0^x\sqrt{\frac{q}{p}}\,dx\right)\right]$$

$$w\big|_{x=0} = 0 \implies \alpha = 0$$

$$w\big|_{x=1} = 0 \implies \sin\left(\lambda \int_0^1 \sqrt{\frac{q}{p}}\,dx\right) = 0 \implies \lambda \int_0^1 \sqrt{\frac{q}{p}}\,dx = n\pi$$

3. Note p, q const & $y = e^{rx} \implies \varepsilon^2 r^2 + \varepsilon^{3/2} p r + q = 0 \implies$

$$r_\pm = \frac{1}{2\varepsilon^2}\Big(-p\varepsilon^{3/2} \pm \underbrace{\sqrt{p^2\varepsilon^3 - 4q\varepsilon^2}}_{\sim \varepsilon\sqrt{-4q}}\Big) \sim \frac{r_0}{\varepsilon} + \frac{r_1}{\varepsilon^{1/2}} + \cdots$$

So, assuming
$$y \sim e^{\frac{\theta}{\varepsilon} + \frac{\varphi}{\varepsilon^{1/2}}}\left(y_0 + \varepsilon^{1/2} y_1 + \cdots\right)$$

$\implies$

$$y' \sim \frac{\theta'}{\varepsilon} y_0 + \frac{1}{\varepsilon^{1/2}}\left(\theta' y_1 + \varphi' y_0\right)$$

$$y'' \sim \frac{\theta_x^2}{\varepsilon^2} y_0 + \varepsilon^{-3/2}\left(\theta_x^2 y_1 + 2\theta'\varphi' y_0\right) + \frac{1}{\varepsilon}\left(\theta_x^2 y_2 + 2\theta'\varphi' y_1 + \varphi_x^2 y_0 + \theta'' y_0 + 2\theta' y_0'\right) + \cdots$$

$O(1)$ $\quad \theta_x^2 = -q \implies \theta_x = \pm i\sqrt{q} \implies \theta = \pm i \int^x \sqrt{q}$

$O(\varepsilon^{1/2})$ $\quad 2\theta'\varphi' y_0 + p\theta' y_0 = 0 \implies \varphi' = -\frac{1}{2} p \implies \varphi = -\frac{1}{2}\int^x p$

$O(\varepsilon)$ $\quad 2\theta' y_0' + \left(\varphi_x^2 + \theta'' + p\varphi'\right) y_0 = 0$

$$y_0' = -\left(\frac{\theta''}{2\theta'} + \frac{p\varphi' + \varphi_x^2}{2\theta'}\right) y_0 = -\left(\frac{\theta''}{2\theta'} - \frac{1}{8}\frac{p^2}{\theta'}\right) y_0$$

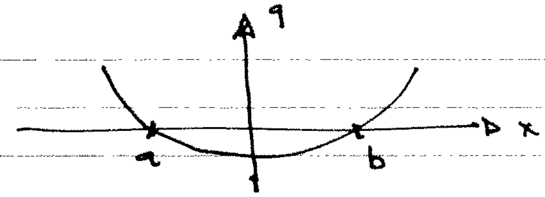
$\implies$

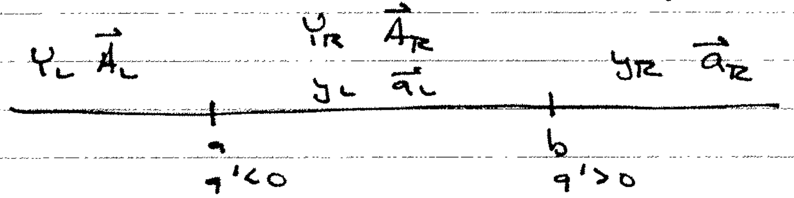
$$y_0 = \frac{\alpha}{\sqrt{\theta_x}} \exp\left(\frac{1}{8}\int \frac{p^2}{\theta'}\right) = \frac{\alpha}{\sqrt{\theta_x}} \exp\left(\mp i \frac{1}{8}\int \frac{p^2}{\sqrt{q}}\right)$$

$\therefore$

$$y \sim \frac{1}{q^{1/4}}\left[\alpha e^{\frac{1}{\varepsilon} i \int \sqrt{q} - \frac{1}{2\sqrt{\varepsilon}}\int p - \frac{1}{8} i \int p^2/\sqrt{q}} + \beta e^{-\frac{1}{\varepsilon} i \int \sqrt{q} - \frac{1}{2\sqrt{\varepsilon}}\int p + \frac{1}{8} i \int \frac{p^2}{\sqrt{q}}}\right]$$

4. $\varepsilon^2 \psi'' = q\psi$ $\quad q = V - E$





Bddness $\Rightarrow$ $A_L = 0$ * $b_R = 0$

$$\vec{A}_R = N \vec{A}_L \qquad \vec{a}_L = M \vec{a}_R$$

$\Rightarrow$

$$A_R = iB_L \qquad a_L = a_R$$

$$A_L = 0 \quad B_R = B_L \qquad b_R = 0 \quad b_L = i a_R$$

Equality: $y_R = y_L \Rightarrow \quad a_L = A_R e^{\phi} \qquad b_L = B_R e^{-\phi}$

$$\phi = \frac{1}{\varepsilon}\int_a^b \sqrt{q}$$

$\therefore$ $A_R = iB_L = iB_R$, $\quad b_L = i a_R = i a_L$ $\Rightarrow$ $a_L = iB_R e^{\phi}$, $\quad a_L = -iB_R e^{-\phi}$

$\Rightarrow$

$$i e^{\phi} = -i e^{-\phi}$$

$\Rightarrow$

$$e^{2\phi} = -1 \qquad \phi = i \cdot \frac{1}{\varepsilon}\int_a^b \sqrt{E - V}\, dx$$

$\Rightarrow$

$$\cos\left(\frac{2}{\varepsilon}\int_a^b \sqrt{E - V}\right) = -1$$

$\Rightarrow$

$$\int_a^b \sqrt{E - V}\, dx = \varepsilon \cdot \pi\left(n + \frac{1}{2}\right)$$

5. $\phi = e^{i\omega t} y(x) \Rightarrow \quad y'' + (\ln A)' y' = -\omega^2 y$

$y \sim e^{\omega^{\alpha}\theta(x)}(y_0 + \omega^{-\alpha} y_1 + \cdots) \Rightarrow$

$O(\omega^2)$ $\quad \theta_x^2 = -1 \Rightarrow \theta = \pm i x$

$O(\omega)$ $\quad \theta'' y_0 + 2\theta' y_0' + (\ln A)'\theta' y_0 = 0 \Rightarrow y_0' = -\left(\frac{\theta''}{2\theta'} + \frac{1}{2}(\ln A)'\right) y_0$

$\Rightarrow y_0 = \alpha / \sqrt{\theta_x A} \Rightarrow \phi \sim \frac{1}{\sqrt{A(x)}}\left[\alpha e^{i(\omega t - x)} + \beta e^{i(\omega t + x)}\right]$