Solutions to HW4

1 a) $t_1 = t$, $t_2 = \varepsilon t \Rightarrow d_t \to \partial_1 + \varepsilon\partial_2 \Rightarrow \quad y \sim y_0 + \varepsilon y_1 + \cdots$

$$(\partial_1^2 + 2\varepsilon\partial_1\partial_2 + \varepsilon^2\partial_2^2)(y_0 + \varepsilon y_1 + \cdots) + y_0 + \varepsilon y_1 + \cdots + \varepsilon y_0^3 + \cdots = 0$$

$O(1)$ $\partial_1^2 y_0 + y_0 = 0 \qquad \Rightarrow \qquad y_0 = A(t_2)\cos[t_1 + \theta(t_2)]$

$y_0(0,0) = a$ & $\partial_1 y_0(0,0) = 0 \qquad A(0)\cos\theta(0) = a$ & $A(0)\sin\theta(0) = 0$

$O(\varepsilon)$ $\partial_1^2 y_1 + y_1 = -2\partial_1\partial_2 y_0 - y_0^3$

$= 2\,(A'\sin + A\theta'\cos) - A^3 \cdot \frac{1}{4}\left[3\cos() + \cos 3()\right]$

sin: $2A' = 0 \Rightarrow A = \text{const} = A_0$

cos: $2A\theta' - \frac{3}{4}A^3 = 0 \quad \Rightarrow \theta' = \frac{3}{8}A_0^2 \Rightarrow \theta = \frac{3}{8}A_0^2 t_2 + \theta_0$

$y_0 = A_0\cos[t_1 + \frac{3}{8}A_0^2 t_2 + \theta_0]$ & ICs $\Rightarrow A_0 = a$, $\theta_0 = 0$

$= a\cos[(1 + \frac{3}{8}a^2\varepsilon)t]$

b) $y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \quad \Rightarrow$

$$y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots + y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots + \varepsilon(y_0^2 + 2\varepsilon y_0 y_1 + \cdots) = 0$$

$O(1)$ $y_0'' + y_0 = 0 \qquad \Rightarrow \qquad y_0 = a\cos t$

$y_0(0) = a$, $y_0'(0) = 0$

$O(\varepsilon)$ $y_1'' + y_1 = -y_0^2 = -\frac{1}{2}a^2[1 + \cos 2t]$

$y_1 = A\cos t + B\sin t - \frac{1}{2}a^2 + \frac{1}{6}a^2\cos 2t$ } no secular terms

$O(\varepsilon^2)$ $y_2'' + y_2 = -2y_0y_1 = -2a\cdot\cos t\cdot\left[A\cos t + B\sin t - \frac{1}{2}a^2 + \frac{1}{6}a^2\cos 2t\right]$

will produce a secular term

c) $t_1 = t$, $t_2 = \varepsilon^2 t$ & $y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \Rightarrow$

$$(\partial_1^2 + 2\varepsilon^2\partial_1\partial_2 + \varepsilon^4\partial_2^2)(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots) + y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

$$+ \varepsilon(y_0^2 + 2\varepsilon y_0 y_1 + \cdots) = 0$$

O(1) $\partial_1^2 y_0 + y_0 = 0$ $\Rightarrow$ $y_0 = A_0(t_2)\cos[t_1 + \theta_0(t_2)]$

$y_0(0,0) = a,\ \partial_1 y_0(0,0) = 0$ $\quad$ $A_0(0)\cos\theta_0(0) = a$, $\quad \sin\theta_0(0) = 0$

O(ε) $\partial_1^2 y_1 + y_1 = -y_0^2 = -A_0^2 \cdot \frac{1}{2}[1 + \cos 2(t_1 + \theta_0)]$

$$y_1 = A_1(t_2)\cos[t_1 + \theta_1(t_2)] - \frac{1}{2}A_0^2 + \frac{1}{6}A_0^2\cos 2(t_1+\theta_0)$$

O(ε²) $\partial_1^2 y_2 + y_2 = -2y_0y_1 - 2\partial_1\partial_2 y_0$

$$= 2(A_0'\sin_0 + A_0\theta_0'\cos_0) - 2A_0\cos_0\left[A_1 c_1 - \frac{1}{2}A_0^2 + \frac{1}{6}A_0^2\cos 2(t_1+\theta_0)\right]$$

$$= 2(A_0' s_0 + A_0\theta_0' \cdot c_0) + A_0^3 c_0 - \frac{1}{3}A_0^3\left[\frac{1}{2}\cos(t_1+\theta_0) + \frac{1}{2}\cos 3(\ )\right]$$
+ stuff

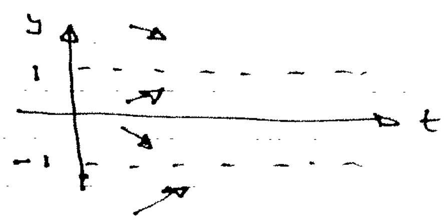
sin: $A_0' = 0 \Rightarrow A_0 = \text{const}$

cos: $2A_0\theta_0' + A_0^3 - \frac{1}{6}A_0^3 = 0 \Rightarrow \theta_0' = -\frac{5}{12}A_0^2 \Rightarrow \theta_0 = -\frac{5}{12}A_0^2 t_2 + \varphi_0$

$$y_0 = A_0\cos\left[t_1 - \frac{5}{12}A_0^2 t_2 + \varphi_0\right]$$ ICs $\Rightarrow A_0 = a$, $\varphi_0 = 0$

$$= a\cdot\cos\left[\left(1 - \frac{5}{12}a^2\varepsilon^2\right)t\right]$$

2. a) $y' = y(1-y^2)$



$y(0) = \varepsilon > 0 \Rightarrow$ $y \to 1$ as $t \to \infty$

b) $y \sim \varepsilon y_0 + \varepsilon^3 y_1 + \cdots \Rightarrow \varepsilon y_0' + \varepsilon^3 y_1' + \cdots = \varepsilon y_0 + \varepsilon^3 y_1 + \cdots - (\varepsilon^3 y_0^3 + \cdots)$

O(ε) $y_0' = y_0$, $y_0(0) = 1$ $\Rightarrow$ $y_0 = e^t$

O(ε³) $y_1' = y_1 - y_0^3 = y_1 - e^{3t}$

$\Rightarrow$ $y_1 = -\frac{1}{2}e^{3t} + Be^t$, $y_1(0) = 0$

$$= \frac{1}{2}(e^t - e^{3t}) = \frac{1}{2}e^t\cdot(1 - e^{2t})$$

$\Rightarrow$ $y \sim \varepsilon e^t \cdot \left[1 + \frac{1}{2}\varepsilon^2(1 - e^{2t})\right]$

← secular terms – grows exponentially – not algebraically

c) $t_1 = t,\ t_2 = \varepsilon^\alpha f(t) \Rightarrow \partial_t \to \partial_1 + \varepsilon^\alpha f' \cdot \partial_2$

$y \sim \varepsilon y_0 + \varepsilon^3 y_1 + \cdots \Rightarrow$

$$(\partial_1 + \varepsilon^\alpha f' \partial_2)(\varepsilon y_0 + \varepsilon^3 y_1 + \cdots) = \varepsilon y_0 + \varepsilon^3 y_1 + \cdots - (\varepsilon^3 y_0 + \cdots)$$

$O(\varepsilon)$ $\partial_1 y_0 = y_0$, $y_0(0,0) = 1$ $\Rightarrow$ $y_0 = A(t_2)e^{t_1}$, $A(0) = 1$

$O(\varepsilon^3)$ $\partial_1 y_1 + \underbrace{f' \cdot \partial_2 y_0}_{f' \cdot A' e^{t_1}} = y_1 - y_0^3 = y_1 - A^3 e^{3t_1}$

want $f' \partial_2 y_2 = -A^3 e^{3t_1} \Rightarrow f' = e^{2t} \Rightarrow f = \frac{1}{2}[e^{2t} - 1]$

$\Rightarrow$ $A' = -A^3 \Rightarrow -\frac{dA}{A^3} = dt_2 \Rightarrow \frac{1}{2A^2} = t_2 + B$

$A = \frac{\pm 1}{\sqrt{2(t_2 + B)}}$ & $A(0) = 1 > 0 \Rightarrow +$ & $B = \frac{1}{2}$

$\Rightarrow$ $y_0 = \frac{1}{\sqrt{1 + 2t_2}} e^{t_1}$

$$= \frac{e^t}{\sqrt{1 + \varepsilon^2(e^{2t} - 1)}}$$

3 b) $m_1 y_1'' + k_1 y_1 = \varepsilon f$

$m_2 y_2'' + k_2 y_2 = -\varepsilon f$

$\frac{1}{k_0} f' + \frac{1}{c_0} f = y_2' - y_1'$

$t_1 = t,\ t_2 = \varepsilon t \Rightarrow \partial_t = \partial_1 + \varepsilon \partial_2$

$y_1 \sim Y_1 + \varepsilon Z_1$ & $y_2 \sim Y_2 + \varepsilon Z_2$ & $f \sim f_0 + \cdots$

$\Rightarrow$

$[m_1(\partial_1^2 + 2\varepsilon \partial_1 \partial_2 + \varepsilon^2 \partial_2^2) + k_1](Y_1 + \varepsilon Z_1 + \cdots) = \varepsilon f_0 + \cdots$

$[m_2(\partial_1^2 + 2\varepsilon \partial_1 \partial_2 + \varepsilon^2 \partial_2^2) + k_2](Y_2 + \varepsilon Z_2 + \cdots) = -\varepsilon f_0$

$\frac{1}{k_0} f_0' + \frac{1}{c_0} f_0 = \partial_1 (Y_2 - Y_1)$

$O(1)\quad (m_i\partial_1^2 + k_i)Y_i = 0 \;\Rightarrow\; Y_i = A_i\cos[\omega_i t_1 + \theta_i] \qquad i=1,2$

$$\omega_i = \sqrt{\frac{k_i}{m_i}}$$

$$\frac{1}{k_0}\partial_1 f_0 + \frac{1}{c_0}f_0 = \partial_1\left[A_2\cos(\omega_2 t_1+\theta_2) - A_1\cos(\omega_1 t_1+\theta_1)\right]$$

$$= -\omega_2 A_2\sin(\omega_2 t_1+\theta_2) + \omega_1 A_1\sin(\omega_1 t_1+\theta_1)$$

$$f_p = \alpha_1\sin(\omega_1 t_1+\theta_1) + \beta_1\cos(\omega_1 t_1+\theta_1) + \alpha_2\sin(\omega_2 t_1+\theta_2) + \beta_2\cos(\omega_2 t_1+\theta_2)$$

$$\boxed{\begin{array}{l}\Omega_1 = \omega_1 t_1+\theta_1\\ \Omega_2 = \omega_2 t_1+\theta_2\end{array}}$$

$\Rightarrow$

$$\frac{1}{k_0}\left[\omega_1\alpha_1\cos_1 - \omega_1\beta_1\sin_1 + \omega_2\alpha_2\cos_2 - \omega_2\beta_2\sin_2\right]$$

$$+\frac{1}{c_0}\left(\alpha_1\sin_1 + \beta_1 c_1 + \alpha_2 s_2 + \beta_2 c_2\right) = -\omega_2 A_2 s_2 + \omega_1 A_1 s_1$$

$c_1:\quad \frac{1}{k_0}\cdot\omega_1\alpha_1 + \frac{1}{c_0}\beta_1 = 0 \qquad \Rightarrow \qquad \beta_1 = -\alpha_1\cdot\frac{\omega_1 c_0}{k_0}$

$s_1:\quad \frac{1}{k_0}(-\omega_1\beta_1) + \frac{1}{c_0}\cdot\alpha_1 = \omega_1 A_1$

$\Rightarrow$

$$\alpha_1 = \frac{\omega_1 A_1}{\frac{1}{c_0} + \frac{\omega_1}{k_0}\cdot\frac{\omega_1 c_0}{k_0}} = \bar{\alpha}_1 A_1 \qquad \bar{\alpha}_1 = \frac{c_0\omega_1}{1+\left(\frac{\omega_1 c_0}{k_0}\right)^2}$$

$$\beta_1 = -\bar{\beta}_1 A_1 \qquad \bar{\beta}_1 = \frac{\omega_1 c_0}{k_0}\cdot\bar{\alpha}_1$$

Similarly

$$\alpha_2 = -\bar{\alpha}_2 A_2 \quad \& \quad \beta_2 = \bar{\beta}_2 A_2$$

$$\therefore\quad f_0 = f_p + \gamma e^{-k_0 t_2/c_0}$$

Note $\quad \partial_1\partial_2 Y_1 = -\omega_1 A_1'\sin\Omega_1 - \omega_1 A_1\theta_1'\cos\Omega_1$

$$\partial_1\partial_2 Y_2 = -\omega_2 A_2'\sin\Omega_2 - \omega_2 A_2\theta_2'\cos\Omega_2$$

$O(2)\quad (m_1\partial_1^2 + k_1)Z_1 + 2m_1\partial_1\partial_2 Y_1 = f_0$

$$(m_2\partial_1^2 + k_2)Z_2 + 2m_2\partial_1\partial_2 Y_2 = -f_0$$

#1: sin: $2m_1(-\omega_1 A_1') = \alpha_1 = \bar{\alpha}_1 A_1$ $\Rightarrow$ $A_1 = A_{10}\, e^{-\lambda_1 t_2}$

cos: $2m_1(-\omega_1 A_1 \theta_1') = \beta_1 = -\bar{\beta}_1 A_1$ $\qquad \lambda_1 = \bar{\alpha}_1 / 2m_1\omega_1$

$$\theta_1' = \frac{\bar{\beta}_1}{2m_1\omega_1} \Rightarrow \theta_1 = \frac{\bar{\beta}_1}{2m_1\omega_1} t_2 + \theta_{10}$$

#2: sin: $2m_2(-\omega_2 A_2') = -\alpha_2 = \bar{\alpha}_2 A_2 \Rightarrow A_2 = A_{20}\, e^{-\lambda_2 t_2}$

$$\lambda_2 = \bar{\alpha}_2 / 2m_2\omega_2$$

$$2m_2(-\omega_2 A_2 \theta_2') = -\beta_2 = -\bar{\beta}_2 A_2$$

$\Rightarrow$

$$\theta_2' = \frac{\bar{\beta}_2}{2m_2\omega_2} \Rightarrow \theta_2 = \frac{\bar{\beta}_2}{2m_2\omega_2} t_2 + \theta_{20}$$

$\therefore$

$$y_1 \sim A_{10}\, e^{-\lambda_1 t_2} \cos[\omega_1 t_1 + \kappa_1 t_2 + \theta_{10}]$$

$$y_2 \sim A_{20}\, e^{-\lambda_2 t_2} \cos[\omega_2 t_1 + \kappa_2 t_2 + \theta_{20}]$$

$$\lambda_i = \frac{\bar{\alpha}_i}{2m_i\omega_i} \qquad \kappa_i = \frac{\bar{\beta}_i}{2m_i\omega_i}$$

$$\bar{\alpha}_i = \frac{c_0\omega_i}{1 + \left(\frac{\omega_i c_0}{k_0}\right)^2} \qquad \bar{\beta}_i = \frac{\omega_i c_0}{k_0} \cdot \bar{\alpha}_i$$