Solutions to HW2

1.a) $x_n = (1-\alpha)x_{n-1} + \beta x_n$ } linear

$\Rightarrow$

$(1-\beta)x_n = (1-\alpha)x_{n-1}$ $\alpha, \beta \neq 1$

$x_n = \xi x_{n-1}$ where $\xi = \frac{1-\alpha}{1-\beta}$

$\Rightarrow$

$x_n = x_0 \cdot \xi^n$

c) $x_{n+1} = x_n e^{-a x_n}$ } nonlinear if $a \neq 0$

$a=0$: $x_{n+1} = x_n \Rightarrow x_n = x_0$

$a \neq 0$: steady-states: $x_n = \bar{x} \Rightarrow \bar{x} = \bar{x} e^{-a\bar{x}}$

$\Rightarrow \bar{x} = 0$ or $\underbrace{e^{-a\bar{x}} = 1}_{\Rightarrow \bar{x} = 0}$

2.a) $N_{n+1} = \alpha \cdot N_n e^{-\beta N_n}$ for $\alpha, \beta > 0$

$N_n = \bar{N} \Rightarrow \bar{N} = \alpha \bar{N} e^{-\beta \bar{N}}$

$\Rightarrow$

$\bar{N} = 0$ or $1 = \alpha e^{-\beta \bar{N}} \Rightarrow e^{-\beta \bar{N}} = \frac{1}{\alpha}$

$-\beta \bar{N} = \ln(\frac{1}{\alpha})$

$\therefore \bar{N} = \frac{1}{\beta}\ln\alpha$ is a steady-state

$\bar{N} = \frac{\ln 1/\alpha}{-\beta}$

$= \frac{1}{\beta}\ln\alpha$

b) $f(N) = \alpha N e^{-\beta N} \Rightarrow f' = -\alpha\beta N e^{-\beta N} + \alpha e^{-\beta N}$

$N = \frac{1}{\beta}\ln\alpha \Rightarrow f' = \alpha e^{-\beta \cdot \frac{1}{\beta}\ln\alpha} - \alpha\beta \cdot \frac{1}{\beta}\ln\alpha \cdot e^{-\beta \cdot \frac{1}{\beta}\ln\alpha}$

$= 1 - \ln\alpha$

$\therefore \underbrace{|f'(\bar{N})| < 1}_{\text{for stability}} \iff |1 - \ln\alpha| < 1$

3. $$N_{n+1} = \frac{\lambda N_n}{(1+aN_n)^b} \qquad a,b,\lambda > 0$$

a) $N_n = \bar{N} \Rightarrow \quad \bar{N} = \frac{\lambda \bar{N}}{(1+a\bar{N})^b}$

$\Rightarrow$

$\bar{N} = 0$ <u>or</u> $1 = \frac{\lambda}{(1+a\bar{N})^b}$

$\Rightarrow$

$$(1+a\bar{N})^b = \lambda$$
$$1+a\bar{N} = \lambda^{1/b}$$
$\Rightarrow$
$$\left.\bar{N} = \frac{\lambda^{1/b}-1}{a}\right\} \text{ so long as } \lambda > 1$$

b) and c) $\quad f(N) = \frac{\lambda N}{(1+aN)^b}$

$\Rightarrow$
$$f' = \frac{\lambda}{(1+aN)^b} - \lambda N \cdot \frac{ab}{(1+aN)^{b+1}}$$

$\bar{N} = 0$: $\quad f'(\bar{N}) = \lambda \quad \therefore \quad 0<\lambda<1 \Rightarrow$ asy. stable
$\lambda > 1 \Rightarrow$ unstable

$\bar{N} = \frac{1}{a}(\lambda^{1/b}-1)$: $\quad f'(\bar{N}) = 1 - \frac{ab\bar{N}}{\lambda^{1/b}}$

$(1+a\bar{N})^b = \lambda$

$$= 1 - \frac{ab}{\lambda^{1/b}} \cdot \frac{1}{a}\left(\lambda^{1/b} - 1\right)$$
$$= 1 - b\left(1 - \lambda^{-1/b}\right)$$

$$|f'|<1 \iff \underbrace{-1 < f'}_{L} < 1, \quad \underbrace{f' < 1}_{R}$$

L: $-1 < 1 - b(1-\lambda^{-1/b}) \Rightarrow \quad b(1-\lambda^{-\frac{1}{b}}) < 2$

R: $1 - b(1-\lambda^{-1/b}) < 1 \Rightarrow \quad -b\underbrace{(1-\lambda^{-1/b})}_{>0 \text{ if } \lambda>1} < 0$ ✓

$\therefore$ always holds

$\therefore$ for $\lambda > 1$, the steady-state is asy. stable (and is physically relevant) if
$$b(1-\lambda^{-\frac{1}{b}}) < 2$$

4. $x_{n+1} = \frac{a x_n}{1 + x_n + b y_n}$ & $y_{n+1} = \frac{c y_n}{1 + d x_n + y_n}$ $a, b, c, d > 0$

a) For both populations, the pop. increases at rate proportional to current pop – with respective coefficients

$$\frac{a}{1 + x_n + b y_n} \quad \text{and} \quad \frac{c}{1 + d x_n + y_n}.$$

The factors decrease if pop. of the other species increases – which is what is expected for competition (i.e., the rate decrease if pop. of other species increases).

b) $x_n = \bar{x}$ & $y_n = \bar{y}$ $\Rightarrow$ $\bar{x} = \frac{a\bar{x}}{1 + \bar{x} + b\bar{y}}$ & $\bar{y} = \frac{c\bar{y}}{1 + d\bar{x} + \bar{y}}$

$\bar{x} = 0 \Rightarrow \bar{y} = \frac{c\bar{y}}{1 + \bar{y}} \Rightarrow \bar{y} = 0$ or $1 = \frac{c}{1+\bar{y}} \Rightarrow \underbrace{\bar{y} = c - 1}_{\text{requires } c > 1}$

$\bar{x} \neq 0 \Rightarrow 1 = \frac{a}{1 + \bar{x} + b\bar{y}}$ & $\bar{y} = \frac{c\bar{y}}{1 + d\bar{x} + \bar{y}}$

$\bar{y} = 0 \Rightarrow 1 = \frac{a}{1 + \bar{x}} \Rightarrow \bar{x} = a - 1$ } requires $a > 1$

$\bar{y} \neq 0 \Rightarrow 1 = \frac{a}{1 + \bar{x} + b\bar{y}}$ & $1 = \frac{c}{1 + d\bar{x} + \bar{y}}$

$\Rightarrow$

$$\begin{aligned} 1 + \bar{x} + b\bar{y} &= a \\ 1 + d\bar{x} + \bar{y} &= c \end{aligned} \quad \Rightarrow \quad \begin{aligned} \bar{x} + b\bar{y} &= a - 1 \\ d\bar{x} + \bar{y} &= c - 1 \end{aligned}$$

$\bar{y} = c - 1 - d\bar{x} \Rightarrow \bar{x}\left[1 + b(-d)\right] = a - 1 - b(c - 1)$

$\bar{x} \cdot (1 - bd) = a - 1 - b(c-1)$ } requires $bd \neq 1$

$$\Rightarrow \quad \bar{x} = \frac{a - 1 - b(c-1)}{1 - bd} \qquad \bar{y} = \frac{c - 1 - d(a-1)}{1 - bd}$$

Four solutions: $(\bar{x}, \bar{y}) = (0,0), (0, c-1), (a-1, 0)$ and the one given above (which is the nonzero solution)

HW2
#4

As for stability,

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{a(1+by)}{(1+x+by)^2} & \frac{-abx}{(1+x+by)^2} \\ \frac{-cdy}{(1+dx+y)^2} & \frac{c(1+dx)}{(1+dx+y)^2} \end{pmatrix}$$

$\bar{x} = \bar{y} = 0$:

$$J = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

$\therefore$ asy stable if $0 < a < 1$ and $0 < c < 1$
unstable if $a > 1$ or $c > 1$

$a = b = c = d = 2 \Rightarrow \bar{x} = \bar{y} = \frac{1}{3} \Rightarrow J = \begin{pmatrix} 5/6 & -1/3 \\ -1/3 & 5/6 \end{pmatrix}$

$tr\, J = 2 \cdot \frac{5}{6} = 5/3 = \frac{20}{12}$

$det\, J = \frac{5 \cdot 5}{6 \cdot 6} - \frac{1}{9} = \frac{7}{12} < 1$ ✓

$det\, J + 1 = \frac{25}{12} < tr\, J$ $\therefore$ unstable