The Five Steps

Objective: find a numerical algorithm to solve

$$\frac{dy}{dt} = f(t, y), \quad \text{for } \ 0 < t,$$

where

$$y(0) = \alpha.$$

STEP 1. Introduce the time points at which the solution will be computed

$$t_j = jk, \quad \text{for } \ j = 0, 1, 2, \ldots, M,$$

where

$$k = \frac{T}{M}.$$

STEP 2. Evaluate the differential equation at the time point $t = t_j$ to obtain

$$y'(t_j) = f(t_j, y(t_j)).$$

STEP 3. Replace the derivative term in STEP 2 with a finite difference formula using the values of y at one or more of the grid points in a neighborhood of t_j.

Example:

$$y'(t_j) = \frac{y(t_{j+1}) - y(t_j)}{k} + \tau_j$$

gives

$$\frac{y(t_{j+1}) - y(t_j)}{k} + \tau_j = f(t_j, y(t_j))$$

STEP 4. Drop the truncation error τ_j.

Example:

$$y_{j+1} - y_j = kf(t_j, y_j),$$

or equivalently,

$$y_{j+1} = y_j + kf(t_j, y_j), \quad \text{for } \ j = 0, 1, 2, \ldots, M - 1$$

where $y_0 = \alpha$.

STEP 5. Check convergence

Numerical Differentiation

Type	Difference Formula	Truncation Term
Forward	$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + \tau_i$	$\tau_i = -\frac{h}{2} f''(\eta_i)$
Backward	$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + \tau_i$	$\tau_i = \frac{h}{2} f''(\eta_i)$
Centered	$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + \tau_i$	$\tau_i = -\frac{h^2}{6} f'''(\eta_i)$
One-sided	$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + \tau_i$	$\tau_i = \frac{h^2}{3} f'''(\eta_i)$
One-sided	$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h} + \tau_i$	$\tau_i = \frac{h^2}{3} f'''(\eta_i)$
Centered	$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + \tau_i$	$\tau_i = -\frac{h^2}{12} f''''(\eta_i)$

Table 1: Numerical Differentiation Formulas. The points $x_1, x_2, x_3, \ldots$ are equally spaced with stepsize $h = x_{i+1} - x_i$. The point η_i is located somewhere between the left- and rightmost points used in the formula.

IVP Solvers

Methods for solving the differential equation $$\frac{d}{dt}\mathbf{y}(t) = \mathbf{f}(t, \mathbf{y})$$			
Method	**Difference Formula**	τ_j	**Properties**
Euler	$\mathbf{y}_{j+1} = \mathbf{y}_j + k\mathbf{f}_j$	$O(k)$	Explicit; Conditionally A-stable
Backward Euler	$\mathbf{y}_{j+1} = \mathbf{y}_j + k\mathbf{f}_{j+1}$	$O(k)$	Implicit; A-stable
Trapezoidal	$\mathbf{y}_{j+1} = \mathbf{y}_j + \frac{k}{2}(\mathbf{f}_j + \mathbf{f}_{j+1})$	$O(k^2)$	Implicit; A-stable
Heun (RK2)	$\mathbf{y}_{j+1} = \mathbf{y}_j + \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2)$ where $\mathbf{k}_1 = k\mathbf{f}_j$ $\mathbf{k}_2 = k\mathbf{f}(t_{j+1}, \mathbf{y}_j + \mathbf{k}_1)$	$O(k^2)$	Explicit; Conditionally A-stable
Classical Runge–Kutta (RK4)	$\mathbf{y}_{j+1} = \mathbf{y}_j + \frac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)$ where $\mathbf{k}_1 = k\mathbf{f}_j$ $\mathbf{k}_2 = k\mathbf{f}(t_j + \frac{k}{2}, \mathbf{y}_j + \frac{1}{2}\mathbf{k}_1)$ $\mathbf{k}_3 = k\mathbf{f}(t_j + \frac{k}{2}, \mathbf{y}_j + \frac{1}{2}\mathbf{k}_2)$ $\mathbf{k}_4 = k\mathbf{f}(t_{j+1}, \mathbf{y}_j + \mathbf{k}_3)$	$O(k^4)$	Explicit; Conditionally A-stable

Table 2: Finite Difference Methods for Solving an IVP. The points $t_1, t_2, t_3, ...$ are equally spaced with stepsize $k = t_{j+1} - t_j$. Also, $\mathbf{f}_j = \mathbf{f}(t_j, \mathbf{y}_j)$.

Numerical Integration

Rule	Integration Formula
Right Box	$\int_{x_i}^{x_{i+1}} f(x)dx = hf(x_{i+1}) + O(h^2)$
Left Box	$\int_{x_i}^{x_{i+1}} f(x)dx = hf(x_i) + O(h^2)$
Midpoint	$\int_{x_{i-1}}^{x_{i+1}} f(x)dx = 2hf(x_i) + \frac{h^3}{3}f''(\eta_i)$
Trapezoidal	$\int_{x_i}^{x_{i+1}} f(x)dx = \frac{h}{2}\left(f(x_i) + f(x_{i+1})\right) - \frac{h^3}{12}f''(\eta_i)$
Simpson	$\int_{x_{i-1}}^{x_{i+1}} f(x)dx = \frac{h}{3}\left(f(x_{i+1}) + 4f(x_i) + f(x_{i-1})\right) - \frac{h^5}{90}f''''(\eta_i)$

Table 3: Numerical Integration Formulas. The points $x_1, x_2, x_3, ...$ are equally spaced with stepsize $h = x_{i+1} - x_i$. The point η_i is located somewhere within the interval of integration.