Mark H. Holmes
Department of Mathematical Sciences
Rensselaer Polytechnic Institute
Troy, NY 12180-3590

Introduction to Perturbation and Asymptotic Methods

3rd Edition

July 21, 2026

Springer Nature

To Colette, Matthew, and Marianna
A small family with big hearts

Preface

First, let me say hello and welcome to the subject of perturbation methods. For those who may be unfamiliar with the topic, the title can be confusing. The first time I became aware of this was during a family reunion when someone asked what I did as a mathematician. This is not an easy question to answer, but I started by describing how a certain segment of the applied mathematics community is interested in problems that arise from physical phenomena. Examples such as water waves, sound propagation, and the aerodynamics of airplanes were discussed. The difficulty of solving such problems was also described in exaggerated detail. Next came the part about how one generally ends up using a computer to actually find the solution. At this point I editorialized on the limitations of computer solutions and why it is important to derive, if at all possible, accurate approximations of the solution. This led naturally to mentioning asymptotics and perturbation methods. These terms ended the conversation because I was unprepared for their reactions. They were not sure exactly what asymptotics meant but they were quite perplexed about perturbation methods. I tried, unsuccessfully, to explain what it means, but it was not until sometime later that I realized the difficulty. For them, as in *Webster's Collegiate Dictionary*, the first two meanings for the word perturb are "to disturb greatly in mind (disquiet); to throw into confusion (disorder)." Although a cynic might suggest this is indeed appropriate for the subject, the intent is exactly the opposite. For a related comment, see Exercise 4.18(d).

In a nutshell, this book serves as an introduction into how to systematically construct an approximation of the solution of a problem that is otherwise intractable. The methods all rely on there being a parameter in the problem that is relatively small. Such a situation is relatively common in applications, and this is one reason why perturbation methods are a cornerstone of applied mathematics. One of the other cornerstones is scientific computing, and it is interesting that the two subjects have grown up together. However, this is not unexpected given their respective capabilities. When using a computer one is capable of solving problems that are nonlinear, inhomogeneous, and multi-dimensional. Moreover, it is possible to achieve very high accuracy. The drawbacks are that computer solutions do not provide much insight into the physics of the problem (particularly for those who do not have access to the appropriate software or computer), and there is always the question as to whether or not the computed solution is correct. On the other hand, perturbation methods are also capable of dealing with nonlinear, inhomogeneous, and multi-dimensional problems (although not to the same extent as computer-generated solutions). The principal objective when using perturbation methods, at least as far as the author is concerned, is to provide a reasonably accurate expression for the solution. By doing this one is able to derive an understanding of the physics of the problem. Also, one can use the result, in conjunction with the original problem, to obtain more efficient numerical procedures for computing the solution.

The methods covered in the text vary widely in their applicability. The first chapter introduces the fundamental ideas underlying asymptotic approximations. This includes their use in constructing approximate solutions of transcendental equations as well as differential equations. In the second chapter, matched asymptotic expansions are used to analyze problems with layers. Chapter 4 describes a method for dealing with problems with more than one time scale. In Chapter 5, the WKB method for analyzing linear singular perturbation problems is developed, while in Chapter 6 a method for dealing with materials containing disparate spatial scales (e.g., microscopic vs. macroscopic) is discussed. The final chapter examines the topics of multiple solutions and stability.

The mathematical prerequisites for this text include a basic background in differential equations and advanced calculus. In terms of difficulty, the chapters are written so that the first sections are elementary or intermediate, while the later sections are somewhat more advanced. Also, the ideas developed in each chapter are applied to a range of problems, including ordinary differential equations, partial differential equations, and difference equations. Scattered through the exercises are applications to integral equations, integro-differential equations, differential-difference equations, and delay equations. What will not be found is an in-depth discussion of the theory underlying the methods. This aspect of the subject is important and references to the more theoretical work in the area are given in each chapter.

The exercises in each section vary in their complexity. In addition to the more standard textbook problems, an effort has been made to include problems from the research literature. The latter are intended to provide a window into the wide range of areas which use perturbation methods. Solutions to some of the exercises are available and they can be obtained from the author's homepage. Also included is an errata sheet. Those who may want to make a contribution to this file, or have suggestions about the text, can reach the author at holmes@rpi.edu .

I would like to express my gratitude to the many students who took my course in perturbation methods at Rensselaer. They helped me immeasurably in deepening my understanding of the subject and provided much-needed encouragement to write this book. It is a pleasure to acknowledge the suggestions of Jon Bell, Ash Kapila, and Bob O'Malley, who read early versions of the manuscript. I would also like to thank Julian Cole, who first introduced me to

perturbation methods and is still, to this day, showing me what the subject is about.

Troy, New York *Mark H. Holmes*
August, 1994

Preface to Second Edition

It's interesting reading something you wrote fifteen years earlier, not just because of what you wrote but also because of what you did not write. You also realize how the subject has evolved, and that certain topics should be rewritten, and others included. It is for these reasons that this second edition was undertaken. As will be explained in the next paragraph, every section has been edited, many only in minor ways, while others have been completely rewritten. New material has also been added. This includes approximations for weakly coupled oscillators, analysis of problems that involve transcendentally small terms, an expanded discussion of Kummer functions, and metastability. Also, one of the core objectives of this book is to develop the ideas underlying perturbation methods and then demonstrate how they can be used in a wide variety of problems. To provide background on some of these areas, two appendices have been added. One on solving difference equations, and another on delay equations. Finally, a few things have also been removed, and the most prominent is the appendix for the numerical solution of boundary-value problems. The code given in the earlier edition is available at the author's home-page, and it is also discussed at length in Holmes [2007]. Finally, the references have been updated, new exercises have been added, and a few of the original exercises have been modified.

There is an interesting side-note that is worth telling. The first edition was written using a commercial software program. At the time, Springer published their books using LaTeX, so they needed to use a translator program to convert the manuscript. With this, they had to redraw, by hand, all of the figures. They also had to write macros to produce some of the lettering used for balancing equations. The result was very nice. Unfortunately, none of this worked for this edition. The original files can not be used with the current version of the commercial software program, and the converted LaTeX files exist only in fragments. Consequently, this edition was almost written from scratch, with every figure redrawn. With this effort comes a few benefits. One is that the manuscript is now in LaTeX, and should be useable for the foreseeable future. Second, many of the figures were redrawn using MATLAB®, and the codes used for the figures are available at the author's homepage. You might find them useful, either in teaching the class or else in providing insights into how to proceed when working out the homework problems. You might also enjoy the videos available, which show some of the solutions of the time-dependent problems solved in the book. Also, there is an errata page as well as answers to some of the exercises.

I would like to thank those who developed and have maintained TeXShop, a free and very good TeX previewer.

Troy, New York *Mark H. Holmes*
July, 2012

Preface to Third Edition

A major change in this edition is the addition of a chapter on the asymptotic approximation of integrals and sums. Other significant changes include extensive revisions to the first and last chapters.

As with the previous editions, files and information relevant to this edition are available on the author's webpage. The website can be accessed through his Rensselaer faculty page or directly at https://holmesrpi.org/.

Troy, New York *Mark H. Holmes*
June, 2026

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