SYMPLECTIC MAPS FOR THE N-BODY PROBLEM

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ABSTRACT

The mapping method of Wisdom [AJ, 87, 577 (1982)] is generalized to encompass all gravitational n-body problems with a dominant central mass. The method is used to compute the evolution of the outer planets for a billion years. This calculation provides independent numerical confirmation of the result of Sussman & Wisdom [Sci, 241, 433 (1988)] that the motion of the planet Pluto is chaotic.

1. INTRODUCTION

Long-term integrations are playing an increasingly important role in investigations in dynamical astronomy. The reason is twofold. First, numerical exploration is an essential tool in the study of complex dynamical systems which can exhibit chaotic behavior, and there has been a growing realization of the importance of chaotic behavior in dynamical astronomy (see, e.g., Wisdom 1987). Second, there has been a phenomenal increase in the capabilities of computers which is bringing many important problems in dynamical astronomy within reach. In particular, there has recently been considerable interest in the long-term evolution of the solar system. Long-term integrations of the solar system include the outer planet integrations of Cohen *et al.* (1973; 1 Myr), Kinoshita & Nakai (1984; 5 Myr), the first Digital Orrery integration (Applegate *et al.* 1986, 210 Myr), the LONGSTOP work (Roy *et al.* 1988; 100 Myr), the second Digital Orrery integration (Sussman & Wisdom 1988; 845 Myr), and the inner planet integrations of Richardson & Walker (1987; 2 Myr), Applegate *et al.* (1986; 3 Myr), and Quinn *et al.* (1991; 3 Myr). Long-term integrations have already produced startling results. Sussman & Wisdom (1988) found numerical evidence that the motion of the planet Pluto is chaotic, with a remarkably short timescale for exponential divergence of trajectories of only 20 million years. This massive calculation consumed several months of time on the Digital Orrery, a computer built specifically for the job which runs at about a third the speed of a Cray 1. Subsequently, Laskar (1989, 1990), in another massive computation, found numerical evidence that the motion of the inner planets is also chaotic, with a divergence timescale of only 5 million years. However, despite the phenomenal progress in computer technology, computers are still too slow for many important applications. For example, it is very important to test the sensitivity of the results concerning the chaotic character of the motions of the planets to uncertainties in initial conditions and parameters. It is also important to clarify the dynamical mechanisms responsible for the chaotic behavior to confirm that the positive Lyapunov exponents are not subtle numerical artifacts. The necessary calculations and those of many other problems of current interest in dynamical astronomy require orders of magnitude greater computing power than is currently available. Regardless of the speed of computers, better, faster algorithms for investigating the n-body problem are always welcome. This paper presents a new method for studying the long-term evolution of the n-body problem which is an order of magnitude faster than traditional methods of numerical integration. The method is a generalization of the "mapping" method introduced by Wisdom (1982, 1983) to study the motion of asteroids near the 3:1 mean-motion resonance with Jupiter. It is applicable to systems which are dominated by a large central mass such as planetary systems or satellite systems.

The mapping method of Wisdom (1982, 1983) was based on the averaging principle. It was noted that most studies of the long-term evolution of the n-body problem relied on the averaging principle in one way or another. This included both analytical and numerical studies. The intuition behind the averaging method is that rapidly oscillating terms tend to average out and give no net contribution to the evolution, while more slowly varying resonant or secular terms accumulate to give significant contributions to the evolution (see Arnold 1974). The intuition behind the mapping method was just the same: If the rapidly oscillating terms do not contribute significantly to the evolution then replacing them with other rapidly oscillating terms will have no ill effect. To get the mapping the rapidly oscillating terms are chosen so that they sum to give delta functions which can be locally integrated to give explicit equations specifying how the system changes from one step to the next. The mapping method was inspired by Chirikov's use of periodic delta functions to derive a Hamiltonian for the standard map (Chirikov 1979). The time step covered by the map is on the order of the period associated with the high-frequency terms. For the asteroid maps, the basic step was one full Jupiter period. The algebraic simplicity of the 3:1 map and the large step size combined to make it extraordinarily fast, about 1000 times faster than even the numerical averaging routines available at the time (Wisdom 1982). The great speed of the map allowed studies of the resonant asteroid motion over much longer times than were previously possible, and significant new phenomena were discovered. In particular, it was found that there was a large zone of chaotic behavior near the 3:1 resonance and that chaotic trajectories in these zones often displayed a peculiar phenomenon in which the eccentricity could remain at relatively low values for several hundred thousand years and then suddenly jump to much higher values. Over longer intervals of millions of years there were periods of low eccentricity behavior interspersed with bursts of high eccentricity behavior. These bursts in eccentricity were subsequently confirmed in traditional direct integrations of Newton's equations (Wisdom 1983; Murray & Fox 1984; Wisdom 1987), and explained perturbatively (Wisdom 1985a). The high eccentricities attained by the chaotic trajectories help explain the formation of the 3:1 Kirkwood gap (Wisdom 1983), as well as provide a mechanism for transporting meteoritic material directly from the asteroid belt to Earth (Wisdom 1985b; Wetherill 1985). Murray

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cision (which we take to be about 10^{-16}) requires about 45 000 function evaluations per orbit. Of course, higher-order methods need to take significantly fewer steps per orbit. Suppose the relative energy error from truncation can be written $\Delta = C(h/N)^{o+1}$, where h is the step size divided by the orbital period, o is the order, N is the number of function evaluations per step, and C is an error constant. We presume that the error constant in this form is comparable for all of the higher-order methods; the factor of N is just a guess, and works in favor of the higher-order methods. Using this estimate we find that even for the eighth-order method of Yoshida (with 15 function evaluations per step), achieving a relative energy error of order the machine precision requires 400 function evaluations per orbit. Thus even the high-order versions of the simple n-body maps may still be inefficient compared to traditional high accuracy integrators. Of course, the relative inefficiency may be outweighed by a better long-term growth of error. To our knowledge the long-term growth of error for the simple symplectic n-body integrators has not yet been carefully examined, particularly in the "high accuracy" mode of operation where the truncation error is of order the machine precision.

Consider in the same manner the possibility of using the n-body maps described in this paper in the "high accuracy" mode. From a numerical integration point of view, the basic difference between these methods and the simple methods just described is that the error constant in these new maps may be expected to be smaller by about the ratio of the planetary masses to the central mass μ. For our solar system, μ is about 10^{-3}. The number of steps per orbit required to achieve the same truncation error as the simple maps is smaller by a factor of $\mu^{-1/(o+1)}$. For a fourth-order method with $\mu = 10^{-3}$, this factor is only about 4. For the eighth-order method, it is about 2. Considering the fact that the steps in the Kepler-based n-body maps are a little more expensive than those in the simple n-body maps, it is not clear that any advantage is gained by using the maps presented here over the simpler maps, at least in the "high accuracy" mode of operation. However, there may be an advantage to using the Kepler-based maps for orbits with high eccentricity. In this case, the simple n-body maps must take many more steps per orbit to stably and accurately execute the orbit, since the basic Kepler motion must be integrated as well. On the other hand, the n-body maps presented in this paper exactly represent a pure Kepler orbit at any eccentricity. Tests in the circular and elliptic restricted problems indicate the Kepler based n-body maps suffer no significant loss of stability or accuracy at high eccentricity. In this case there may be a significant advantage in using them over the simple maps even in the "high accuracy" mode.

On the other hand, consider the use of the n-body maps introduced in this paper in the "qualitative" mode of operation. Typically, efficient traditional integrators take on the order of 100 steps per orbit. We have found that in solar system integrations the qualitative behavior is reliably reproduced with as few as ten steps per orbit. Such a small number of steps per orbit is stable here because the Kepler motion is represented exactly and does not have to be rediscovered each orbit. The reduction in the number of function evaluations by a factor of 10 accounts roughly for the order of magnitude greater speed of the new mapping method over traditional integrators. The new n-body maps are the clear winners for qualitative studies.

Of course, the relative merits of the various methods in the two different modes of operation should be studied more thoroughly to check the estimates given here.

9. THE OUTER PLANETS FOR A BILLION YEARS

We have carried out numerous tests of the new n-body maps. First, a number of surfaces of section for the circular restricted three-body problem were computed with the new map and compared to sections computed with the conventional Bulirsch–Stoer numerical integration algorithm. The agreement was excellent and provided valuable initial experience with the new maps. These tests demonstrated the reliability and efficiency of the map at high eccentricity. The n-body maps have also been implemented for the planar elliptic restricted three-body problem. The numerical integrations reported in Wisdom (1983), which also used the conventional Bulirsch–Stoer algorithm, were all repeated with the map, with particular attention to whether the map would give the correct diagnosis of whether the trajectory was chaotic or quasiperiodic. In every case, the map agreed with the earlier results. Of course, the jumps in eccentricity were also recovered. Note that the codes for the various versions of the restricted three-body problem can be written to take advantage of the known fixed orbit of the two massive bodies. Rather than present these initial tests in detail, we present a much more stringent test. We have used the map to compute the evolution of the outer planets, including Pluto as a test particle, for about 1.1 billion years. For this problem the evolution has already been computed for 845 million years using conventional integration techniques on the Digital Orrery (Sussman & Wisdom 1988), and comparison can be made to those results.

We have chosen to use the second-order version of the mapping, which optimizes the agreement of the mapping Hamiltonian with the true Hamiltonian in accordance with our original motivation based on the averaging principle. We have used the exact form of the interaction Hamiltonian, and Pluto is given a Jacobi index below those of the massive planets. Of course, in order to make comparisons the initial conditions and parameters must be the same as those used in the Digital Orrery integrations (Applegate *et al.* 1986). The only parameter left to choose is the step size, or mapping period. The map is used in the "qualitative" mode and the step size is chosen to be relatively large. A number of preliminary tests indicate that the map does not work well for this problem if fewer than five steps are taken per Jupiter orbit period, which is about 12 yr. To add a margin of safety, a step size of 1 yr was chosen. This may be compared to typical steps of 40 days or less that have been used in other studies of the outer planets using conventional numerical integration techniques. The relative energy error oscillates as expected, and, using this step size, has a rather large peak to peak amplitude of about 10^{-5}. The map is remarkably fast. A billion year evolution of the outer planets takes only 14 days on a Hewlett-Packard HP9000/835 RISC workstation.

All of the principal results of Sussman & Wisdom (1988) are reproduced in the mapping evolution. For example, the argument of perihelion of Pluto again displays a 34 million yr modulation. The quantity $h = e \sin \varpi$, where ϖ is the longitude of perihelion, displays its strong 137 million yr period. This is illustrated in Fig. 1 which is to be compared with h as computed using the Stormer multistep predictor on the Digital Orrery, shown in Fig. 2. The two plots are not identical, but the similarity is astounding. The inclination of Pluto