Chapter 4

Linear Systems

This chapter, and the one that follows, consider problems that involve two or more first-order ordinary differential equations. Together the equations form what is called a first-order system. These are very common. To explain why, it is worth considering a couple of examples.

Example 1: Mechanics

As stated on several occasions earlier in this text, one of the biggest generators of differential equations is Newton's second law, which states that $F = ma$. To demonstrate its connection with a system of differential equations, let $x(t)$ denote the position of an object. The velocity is then $v = x'(t)$, and the acceleration is $a = x''(t)$. So, $F = ma$ can be written as $mv' = F$. Along with the equation $x' = v$, the resulting system is

$$\frac{dx}{dt} = v,$$
$$\frac{dv}{dt} = \frac{1}{m}F.$$

As an example, for a uniform gravitation field, and including air resistance, then $F = -mg - cv$ (see Section 2.3.2). In this case, the system becomes

$$x' = v,$$
$$v' = -g - \frac{c}{m}v.$$

This is a linear first-order system for x and v. It is also inhomogeneous since $x \equiv 0$ and $v \equiv 0$ is not a solution. ■

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Example 2: Epidemics

Epidemics, such as the black death, COVID-19, and cholera, have come and gone throughout human history. Given the catastrophic nature of these events there is a long history of scientific study trying to predict how and why they occur. One of particular prominence is the Kermack-McKendrick model for epidemics. This assumes the population can be separated into three groups. One is the population $S(t)$ of those susceptible to the disease, another is the population $I(t)$ that is ill, and the third is the population $R(t)$ of individuals that have recovered. A model that accounts for the susceptible group getting sick, the subsequent increase in the ill population, and the eventual increase in the recovered population is the following set of equations [Holmes, 2019]

$$\begin{aligned}\frac{dS}{dt} &= -k_1 SI,\\ \frac{dI}{dt} &= -k_2 I + k_1 SI,\\ \frac{dR}{dt} &= k_2 I.\end{aligned}$$

Given the three groups, and the letters used to designate them, this is an example of what is known as a SIR model in mathematical epidemiology. For us, this is an example of a nonlinear first-order system for S, I, and R. The reason it is nonlinear is the SI term that appears in the first two equations.

As you might expect, solving a nonlinear system can be challenging. So, in this chapter, we will concentrate on linear systems. In the next chapter, nonlinear problems are considered.

4.1 ▪ Linear Systems

To get things started, consider the problem of solving

$$x' = ax + by, \tag{4.1}$$
$$y' = cx + dy. \tag{4.2}$$

This is a first-order, linear, homogeneous system. In these equations, $x(t)$ and $y(t)$ are the dependent variables, and a, b, c, and d are constants. This can be written in system form as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

A simpler way to write this is as

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{4.3}$$

where the vector is

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix},$$

and the matrix is

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.4}$$

The equation in (4.3) plays a central role throughout this chapter. Written in this way, we could be dealing with 20 equations, or 200 equations, and not just the two in (4.1) and (4.2).

For those a bit rusty on the basic rules for working with matrices and vectors, a short summary is provided in Appendix A.

Before getting into the discussion of how to solve (4.3), it is worth considering what we already know about the solution.

4.1.1 ▪ Example: Transforming to System Form

In Section 3.5, Example 1, we found that for

$$y'' + 2y' - 3y = 0 \tag{4.5}$$

the roots of the characteristic equation are $r_1 = -3$ and $r_2 = 1$. The resulting independent solutions are $y_1 = e^{-3t}$ and $y_2 = e^t$. In this example, the differential equation, along with its solutions, are translated into vector form.

a) Write (4.5) as a linear first-order system as in (4.3).

The standard way to do this is to let $v = y'$, so the differential equation can be written as $v' + 2v - 3y = 0$, or equivalently, $v' = 3y - 2v$. This, along with the equation $y' = v$, gives us the system

$$\begin{aligned} y' &= v, \\ v' &= 3y - 2v. \end{aligned}$$

In other words, we have an equation of the form (4.3), where

$$\mathbf{x} = \begin{pmatrix} y \\ v \end{pmatrix}, \qquad \text{and} \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}.$$

b) Write the two linearly independent solutions in vector form.

For $y_1 = e^{-3t}$, then $v_1 = y_1' = -3e^{-3t}$. Letting $\mathbf{x}_1$ be the solution vector coming from y_1, then

$$\mathbf{x}_1 = \begin{pmatrix} y_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} e^{-3t} \\ -3e^{-3t} \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t} = \mathbf{a}_1 e^{r_1 t},$$

where $r_1 = -3$ and

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Similarly, since $v_2 = y_2' = e^t$, then letting $\mathbf{x}_2$ be the vector version of y_2,

$$\mathbf{x}_2 = \begin{pmatrix} y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t = \mathbf{a}_2 e^{r_2 t},$$

where $r_2 = 1$ and

$$\mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

c) Write the general solution in vector form.

The general solution for the second-order equation is $y = c_1y_1 + c_2y_2$. From this, we get that $v = y' = c_1y_1' + c_2y_2'$. Therefore, the general solution vector is

$$\begin{aligned} \mathbf{x} = \begin{pmatrix} y \\ v \end{pmatrix} &= \begin{pmatrix} c_1y_1 + c_2y_2 \\ c_1y_1' + c_2y_2' \end{pmatrix} = \begin{pmatrix} c_1y_1 \\ c_1y_1' \end{pmatrix} + \begin{pmatrix} c_2y_2 \\ c_2y_2' \end{pmatrix} \\ &= c_1\mathbf{x}_1 + c_2\mathbf{x}_2. \quad \blacksquare \end{aligned} \tag{4.6}$$

A very useful observation to make about the above example is that the linearly independent solutions have the form $\mathbf{x} = \mathbf{a}e^{rt}$, where $\mathbf{a}$ is a constant vector. In fact, when the time comes to solve (4.3) we will simply assume that $\mathbf{x} = \mathbf{a}e^{rt}$, and then find r and $\mathbf{a}$. Also, note that for the single linear equation $x' = ax$, there is one linearly independent solution. As the above example shows, for two linear first-order equations there are two linearly independent solutions. Consequently, it should not be a surprise to find out that for n linear first-order equations there are n linearly independent solutions.

4.1.2 ▪ General Version

We are going to consider solving homogeneous linear first-order systems. Assuming there are n dependent variables, then the system can be written as

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n, \end{aligned}$$

where the a_{ij}'s are constants. This can be written as

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{4.7}$$

where $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{x}$ is an n-vector, given, respectively, as

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

For an initial value problem, an n-vector $\mathbf{x}_0$ would be given, and the condition to be satisfied would be $\mathbf{x}(0) = \mathbf{x}_0$.

Because (4.7) is linear and homogeneous, the principle of superposition holds (see page 5). Therefore, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of (4.7), then

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

is a solution for any values of the constants c_1 and c_2.

As a final comment, the inhomogeneous equation $\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f}$ is not considered in this chapter, but it is considered in Section 6.8.

Exercises

1. Write the following as $\mathbf{x}' = \mathbf{A}\mathbf{x}$, making sure to identify the entries in $\mathbf{x}$ and $\mathbf{A}$. If initial conditions are given, write them as $\mathbf{x}(0) = \mathbf{x}_0$.

   a) $u' = u - v$
      $v' = 2u - 3v$

   b) $2u' = -u$
      $3v' = u + v$

   c) $x' = x - y + 2z$
      $y' = x$
      $z' = -x + 5y$

   d) $u' = u - v$
      $v' = 2u - 3v$
      $u(0) = -1$, $v(0) = 0$

   e) $x' = 2x - z$
      $y' = x + y + z$
      $3z' = 2y + 6z$
      $x(0) = -1$, $y(0) = 0$, $z(0) = 3$

2. For the following: i) Write the equation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$. ii) Find the general solution of the second-order equation and then write it in vector form as $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$, where $\mathbf{x}_1 = \mathbf{a}_1 e^{r_1 t}$ and $\mathbf{x}_2 = \mathbf{a}_2 e^{r_2 t}$. Make sure to identify $\mathbf{a}_1$, $\mathbf{a}_2$, r_1 and r_2.

   a) $y'' + 2y' - 3y = 0$

   b) $4y'' + y = 0$

   c) $4u'' + 3u' - u = 0$

   d) $u'' + 4u' = 0$

3. Show that the given vector $\mathbf{x}$ is a solution of the differential equation. Also, what initial condition does $\mathbf{x}$ satisfy?

   a) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}$

   b) $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-3t}$

   c) $\mathbf{x}' = \begin{pmatrix} \frac{1}{2} & 1 \\ -1 & \frac{1}{2} \end{pmatrix}\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{t/2}$

4. The vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the given differential equation. Show that $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is a solution no matter what the values of c_1 and c_2.

   a) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t}$

   b) $\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{t}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

5. This problem considers some of the connections between a second-order equation and a first-order system.

   a) Assuming that $c \neq 0$, show that (4.1), (4.2) can be reduced to the second-order linear equation

   $$y'' - (a+d)y' + (ad - bc)y = 0.$$

   b) Using the result from part (a), transform $y'' + 2y' - 3y = 0$ into a first-order system where none of the entries in $\mathbf{A}$ are zero.

   c) Using part (a), and the example in Section 4.1.1, find the general solution of the differential equation in Exercise 3(a).

4.2 ▪ General Solution of a Homogeneous Equation

The problem considered here is

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}, \text{ for } t > 0. \tag{4.8}$$

From (4.6), as well as Exercise 2 in the previous section, we have an idea of what the general solution of this equation looks like. Namely, if we are able to find n linearly independent solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, …, $\mathbf{x}_n(t)$, then the general solution can be written as

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t), \tag{4.9}$$

where c_1, c_2, …, c_n are arbitrary constants.

The requirement to be linearly independent is a simple generalization of the definition given in Section 3.2. Namely, $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, …, $\mathbf{x}_n(t)$ are **linearly independent** if, and only if, the only constants c_1, c_2, …, c_n that satisfy

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}, \quad \forall\, t \geq 0, \tag{4.10}$$

are $c_1 = 0$, $c_2 = 0$, …, $c_n = 0$. In the above equation, $\mathbf{0}$ is the **zero vector**, which means that all of its components are zero. Also, the symbol $\forall$ is a mathematical shorthand for "for all" or "for every."

In the last chapter the Wronskian was used to determine independence. It is possible to also use the Wronskian with (4.8), but this is not particularly useful for larger n. There is an easier way to show independence, and this will be explained in Section 4.4.

The general solution of (4.8) is found by assuming that $\mathbf{x} = \mathbf{a}e^{rt}$, where $\mathbf{a}$ is a constant vector. Differentiating this expression, $\mathbf{x}' = r\,\mathbf{a}e^{rt}$, and so (4.8) becomes $r\,\mathbf{a}e^{rt} = \mathbf{A}(\mathbf{a}e^{rt})$. Since e^{rt} is never zero we can divide by it, which gives us the equation

$$\mathbf{A}\mathbf{a} = r\,\mathbf{a}. \tag{4.11}$$

What we want are nonzero solutions of this equation, and so we require that $\mathbf{a} \neq \mathbf{0}$. This problem for r and $\mathbf{a}$ is called an **eigenvalue problem**, where r is an **eigenvalue**, and $\mathbf{a}$ is an **associated eigenvector**. This is one of the core topics covered in linear algebra. We do not need to know the more theoretical aspects of this problem, but we certainly need to know how to solve it. So, for completeness, the more pertinent aspects of an eigenvalue problem are reviewed next.

It is worth pointing out that it is possible to solve (4.8) without using eigenvalues and eigenvectors, and how this is done is explained in Section 6.8.

4.3 ▪ Review of Eigenvalue Problems

Given an $n \times n$ matrix $\mathbf{A}$, its eigenvalues r and the associated eigenvectors $\mathbf{a}$ are found by solving

$$\mathbf{A}\mathbf{a} = r\,\mathbf{a}. \tag{4.12}$$

It is required that $\mathbf{a}$ is not the zero vector. There are no conditions placed on r, and it can be real or complex valued.

In preparation for solving the above equation, it is first rewritten as $\mathbf{A}\mathbf{a} - r\mathbf{a} = \mathbf{0}$, or equivalently as

$$(\mathbf{A} - r\mathbf{I})\mathbf{a} = \mathbf{0}. \tag{4.13}$$

The $n \times n$ matrix $\mathbf{I}$ is known as the **identity matrix** and it is defined as

$$\mathbf{I} \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

For example, when $n = 2$ and $n = 3$,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In linear algebra it is shown that for the equation (4.13) to have a nonzero solution, it is necessary that the matrix $\mathbf{A} - r\mathbf{I}$ be singular, or non-invertible. What this means is that the determinant of this matrix is zero. This gives rise to the following method for solving the eigenvalue problem.

Eigenvalue Algorithm. *The procedure used to solve the eigenvalue problem consists of two steps:*

1. *Find the r's by solving*

$$\det(\mathbf{A} - r\mathbf{I}) = 0. \tag{4.14}$$

This is known as the ***characteristic equation****, and the left-hand-side of this equation is an nth degree polynomial in r.*

2. *For each eigenvalue r, find the associated eigenvectors by finding the nonzero solutions of*

$$(\mathbf{A} - r\mathbf{I})\mathbf{a} = \mathbf{0}. \tag{4.15}$$

In this textbook we are mostly interested in systems involving two equations. For those who might not remember, the determinant of a 2×2 matrix is defined as

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \equiv a_{11}a_{22} - a_{12}a_{21}.$$

In the second step of the algorithm, when solving (4.15), we are interested in finding the vectors that can be used to form the general solution of this equation. To say this more mathematically, we want to find linearly independent solutions. In n dimensions, it is not possible to have more than n linearly independent vectors. Consequently, n is the maximum number of linearly independent eigenvectors you can find for an $n \times n$ matrix $\mathbf{A}$.

The following examples all involve 2×2 matrices. What is illustrated are the various situations that can arise with eigenvalue problems. In these examples, the eigenvector will be written in component form as

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{4.16}$$

Example 1: Two Real Eigenvalues

For

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

we get that

$$\mathbf{A} - r\mathbf{I} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - r\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-r & 1 \\ 1 & 2-r \end{pmatrix}.$$

Since $\det(\mathbf{A} - r\mathbf{I}) = (2-r)^2 - 1 = r^2 - 4r + 3$, then the characteristic equation (4.14) is $r^2 - 4r + 3 = 0$. Solving this we get that the eigenvalues are $r_1 = 3$ and $r_2 = 1$. For r_1, (4.15) takes the form

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In component form, we have that

$$\begin{aligned} -a + b &= 0, \\ a - b &= 0. \end{aligned}$$

The solution is $b = a$, and so the eigenvectors are

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} = a\,\mathbf{a}_1, \tag{4.17}$$

where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.18}$$

For the second eigenvalue $r_2 = 1$, one finds that the eigenvectors have the form $\mathbf{a} = a\mathbf{a}_2$, where a is an arbitrary nonzero constant and

$$\mathbf{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$