Chapter 3

Second-Order Linear Equations

The general version of the differential equations considered in this chapter can be written as

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t), \tag{3.1}$$

where $p(t)$, $q(t)$, and $f(t)$ are given. One of the reasons this equation gets its own chapter is Newton's second law, which, if you recall, is $F = ma$. To explain, if $y(t)$ is the displacement, then the acceleration is $a = y''$, and this gives us the differential equation $my'' = F$. In this chapter we are considering problems when F is a linear function of velocity y' and displacement y. Later, in Chapter 5, we will consider equations where the dependence is nonlinear. It is because of the connections with the second law that $f(t)$ in (3.1) is often referred to as the **forcing function**.

In the previous chapter, for first-order linear differential equations, we very elegantly derived a formula for the general solution. This will not happen for second-order equations. *All* of the methods derived in this chapter are, in fact, just good, or educated, guesses on what the answer is. There are non-guessing methods, and one example involves using a Taylor series expansion of the solution. An illustration of how this is done can be found in Exercise 8 on page 54.

To use a guessing approach, it becomes essential to know the mathematical requirements for what can be called a general solution. This is where we begin.

Introduction to Differential Equations, M. H. Holmes, 2023

3.1 ▪ Initial Value Problem

A typical initial value problem (IVP) consists of solving (3.1), for $t > 0$, with the initial conditions

$$y(0) = \alpha, \quad \text{and} \quad y'(0) = \beta, \tag{3.2}$$

where α and β are given numbers. Because our solution methods involve guessing, it is important that we know when to stop guessing and conclude we have found the solution. This is why the next result is useful.

Existence and Uniqueness Theorem. *If $p(t)$, $q(t)$, and $f(t)$ are continuous for $t \geq 0$, then there is exactly one smooth function $y(t)$ that satisfies (3.1) and (3.2).*

In stating that $y(t)$ is a smooth function, it is meant that $y''(t)$ is defined and continuous for $t \geq 0$. Those interested in the proof of the above theorem, or the theoretical foundations of the subject, should consult Coddington and Carlson [1997].

So, according to the above theorem, if we find a smooth function that satisfies the differential equation and initial conditions, then that is the solution, and the only solution, of the IVP.

3.2 ▪ General Solution of a Homogeneous Equation

The **associated homogeneous equation** for (3.1) is

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0. \tag{3.3}$$

We need to spend some time discussing what it means to be the general solution of this equation. So, consider Exercise 5(a), in Section 1.2. Assuming you did this exercise, you found that given solutions $y_1 = e^{2t}$ and $y_2 = e^t$ of $y'' - 3y' + 2y = 0$, then

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3.4}$$

is a solution for any value of c_1 and c_2. What is important here is that this is a **general solution** of the differential equation. Roughly speaking, this means that any, and all, solutions of the differential equation are included in this formula. A more precise statement is that, no matter what the values of α and β, there are values for c_1 and c_2 so that (3.4) satisfies the differential equation (3.3) as well as the given initial conditions in (3.2).

This gives rise to the question: what is required so a solution like the one in (3.4) can be claimed to be a general solution? The answer is given in the next result.

General Solution Theorem. *The function* $y(t) = c_1y_1(t) + c_2y_2(t)$, *where* c_1 *and* c_2 *are arbitrary constants, is a general solution of (3.3) if the following are true:*

1. $y_1(t)$ *and* $y_2(t)$ *are solutions of (3.3), and*
2. $y_1(0)y_2'(0) - y_1'(0)y_2(0) \neq 0$.

To explain where these two requirements come from, the first one guarantees that $y(t)$ is a solution of (3.3) no matter what the values of c_1 and c_2. As for the initial conditions (3.2), they require that

$$\begin{aligned} c_1y_1(0) + c_2y_2(0) &= \alpha, \\ c_1y_1'(0) + c_2y_2'(0) &= \beta. \end{aligned}$$

Solving these equations, one gets

$$c_1 = \frac{\alpha y_2'(0) - \beta y_2(0)}{y_1(0)y_2'(0) - y_1'(0)y_2(0)},$$

with a similar expression for c_2. So, as long as $y_1(0)y_2'(0) \neq y_1'(0)y_2(0)$ it is possible to find c_1 and c_2 so the initial conditions are satisfied (no matter what the values of α and β). In other words, $y(t)$ is a general solution.

Example: Show that $y = c_1e^{-3t} + c_2e^t$ is a general solution of $y'' + 2y' - 3y = 0$.

Answer: In this case, $y_1(t) = e^{-3t}$ and $y_2(t) = e^t$. It is not hard to show that they are solutions of the differential equation (see Section 1.2). To check on the second requirement, note that $y_1' = -3e^{-3t}$ and $y_2' = e^t$. So, $y_1(0)y_2'(0) - y_1'(0)y_2(0) = 4 \neq 0$. Therefore, y is a general solution. ■

3.2.1 ▪ Linear Independence and the Wronskian

It is possible to restate the General Solution Theorem given above as: "The function $y(t) = c_1y_1(t) + c_2y_2(t)$, where c_1 and c_2 are arbitrary constants, is a general solution of (3.3) if $y_1(t)$ and $y_2(t)$ are linearly independent solutions of (3.3)." The requirement that y_1 and y_2 are **linearly independent** means that the only constants c_1 and c_2 that satisfy

$$c_1y_1(t) + c_2y_2(t) = 0, \quad \forall\, t \geq 0, \tag{3.5}$$

are $c_1 = 0$ and $c_2 = 0$. This is, effectively, the same definition of linear independence used in linear algebra. The difference is that we have functions rather than vectors. If it is possible to find either $c_1 \neq 0$ or $c_2 \neq 0$ so (3.5) holds, then y_1 and y_2 are said to be **linearly dependent**. Also, in (3.5), the symbol $\forall$ is a mathematical shorthand for "for all" or "for every."

The question arises about how the "independent solutions" version of the theorem is the same as the "$y_1(0)y_2'(0) - y_1'(0)y_2(0) \neq 0$" version. To explain, given two solutions y_1 and y_2 of (3.3), one way to determine if they are independent is to use what is called the Wronskian of y_1 and y_2. This is defined as

$$W(y_1, y_2) \equiv \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}. \tag{3.6}$$

For those unfamiliar with determinants, this can be written as

$$W(y_1, y_2) \equiv y_1 y_2' - y_2 y_1'. \tag{3.7}$$

The usefulness of this function is due, in part, to the next result.

Independent Solutions Test. *If y_1 and y_2 are solutions of (3.3), then y_1 and y_2 are independent if, and only if, $W(y_1, y_2)$ is nonzero.*

The Wronskian comes into this problem because (3.5) must hold on the interval $0 \leq t < \infty$. So, (3.5) can be differentiated, which gives us the equation $c_1y_1' + c_2y_2' = 0$. This, along with (3.5), provides two equations for c_1 and c_2. It is not hard to show that if $W(y_1, y_2) \neq 0$, then the only solution to these two equations is $c_1 = c_2 = 0$. Consequently, if $W(y_1, y_2) \neq 0$, then y_1 and y_2 are independent.

Now, as shown in Exercise 5, either $W(y_1, y_2)$ is identically zero or else it is never zero. Given that $y_1(0)y_2'(0) - y_1'(0)y_2(0)$ is the value of $W(y_1, y_2)$ at $t = 0$, then from the Independent Solutions Test we conclude that $y_1(t)$ and $y_2(t)$ are linearly independent if, and only if, $y_1(0)y_2'(0) - y_1'(0)y_2(0) \neq 0$. So, the two versions of the theorem are equivalent.

Exercises

1. Assuming $\omega \neq 0$, show that $y = c_1e^{\omega t} + c_2e^{-\omega t}$ is a general solution of $y'' - \omega^2 y = 0$.
2. Show $y = c_1e^{-\alpha t} + c_2te^{-\alpha t}$ is a general solution of $y'' + 2\alpha y' + \alpha^2 y = 0$.
3. Assuming $b \neq 0$, show that $y_1 = 1$ and $y_2 = e^{-bt}$ are independent solutions of $y'' + by' = 0$.
4. Assuming $\omega \neq 0$, show that $y_1 = \cos(\omega t)$ and $y_2 = \sin(\omega t)$ are independent solutions of $y'' + \omega^2 y = 0$.
5. If y_1 and y_2 are solutions of (3.3), show that $\frac{d}{dt}W + p(t)W = 0$. Use this to derive Abel's formula, which is that

$$W(y_1, y_2) = W_0 e^{-\int_0^t p(r)dr},$$

where $W_0 = y_1(0)y_2'(0) - y_1'(0)y_2(0)$ is the value of W at $t = 0$.

3.3 ▪ Solving a Homogeneous Equation

The solution of the homogeneous equation

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \tag{3.8}$$

can be found by assuming that $y = e^{rt}$. With this, $y' = re^{rt}$, and $y'' = r^2e^{rt}$, and so (3.8) becomes $(r^2 + br + c)e^{rt} = 0$. Since e^{rt} is never zero, we conclude that

$$r^2 + br + c = 0. \tag{3.9}$$

This is called the **characteristic equation** for (3.8). It is easily solved using the quadratic formula, which gives us that

$$r = \frac{1}{2}\Big(- b \pm \sqrt{b^2 - 4c}\Big). \tag{3.10}$$

There are three possibilities here:

1. there are two real-valued r's: this happens when $b^2 - 4c > 0$,
2. there is one r: this happens when $b^2 - 4c = 0$, and
3. there are two complex-valued r's: this happens when $b^2 - 4c < 0$.

The case of when the roots are complex-valued requires a short introduction to complex variables, and so it is done last.

3.3.1 ▪ Two Real Roots

When there are two real-valued roots, say, r_1 and r_2, then the two corresponding solutions of (3.8) are $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$. It is left as an exercise to show they are independent. Therefore, the resulting general solution of (3.8) is

$$y = c_1e^{r_1 t} + c_2e^{r_2 t}.$$

3.3.2 ▪ One Real Root and Reduction of Order

When there is only one root, the second solution can be found using what is called the *reduction of order method*. To explain, if you know a solution $y_1(t)$, it is possible to find a second solution by assuming that $y_2(t) = w(t)y_1(t)$. In our case, we know that $y_1(t) = e^{rt}$, where $r = -b/2$, is a solution. So, to find a second solution it is assumed that $y(t) = w(t)e^{rt}$. Substituting this into (3.8), and simplifying, yields the differential equation

$$w'' + (2r + b)w' + r^2 + br + c = 0.$$

Since $r = -b/2$, and $4c = b^2$, then the above differential equation reduces to just $w'' = 0$. Integrating this once gives $w' = d_1$ and then integrating again yields $w = d_1 t + d_2$, where d_1 and d_2 are arbitrary constants. With this our second solution is $y = (d_1 t + d_2)e^{rt}$. A solution that is independent of $y_1 = e^{rt}$ is obtained by taking $d_1 = 1$ and $d_2 = 0$, which means that $y_2 = te^{rt}$. The fact that they are independent follows from the Independence Test since $W(y_1, y_2) = 2e^{rt}$ is nonzero. Therefore, the resulting general solution of (3.8) is

$$y = c_1 e^{rt} + c_2 t e^{rt}.$$

3.4 ▪ Complex Roots

An example of a differential equation that generates complex-valued roots is

$$y'' + 4y' + 13y = 0. \tag{3.11}$$

Assuming $y = e^{rt}$, we obtain the characteristic equation $r^2 + 4r + 13 = 0$. The two solutions of this are $r_1 = -2 + 3i$ and $r_2 = -2 - 3i$. Proceeding as in the case of two real-valued roots, the conclusion is that the resulting general solution of (3.11) is

$$\begin{aligned} y &= c_1 e^{r_1 t} + c_2 e^{r_2 t} \\ &= c_1 e^{(-2+3i)t} + c_2 e^{(-2-3i)t}. \end{aligned} \tag{3.12}$$

Because complex numbers are used in the exponents, if this expression is used as the general solution, then c_1 and c_2 must be allowed to also be complex-valued.

Although solutions as in (3.12) are used, particularly in physics, there are other ways to write the solution that do not involve complex numbers. Even if (3.12) is used, there is still the question of how to evaluate an expression such as e^{3i}. For this reason, a short introduction to complex variables is needed.

3.4.1 ▪ Euler's Formula and its Consequences

The key for working with complex exponents is the following formula.

Euler's Formula. *If θ is real-valued then*

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{3.13}$$

It is not possible to overemphasize the importance of this formula. It is one of those fundamental mathematical facts that you must memorize. For those who might wonder how this formula is obtained, it comes from writing down the Maclaurin series of $e^{i\theta}$, $\cos\theta$, and $\sin\theta$, and then showing that they satisfy (3.13).

As it must, (3.13) is consistent with the usual rules involving arithmetic, algebra, and calculus. The examples below provide illustrations of this fact.

Example 1: Since, by definition, $i = \sqrt{-1}$, then $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. Also,

$$\begin{aligned}(a + ib)^2 &= (a + ib)(a + ib) \\ &= a^2 - b^2 + 2iab.\end{aligned}$$

It is useful to be able to identify the real and imaginary part of a complex number. So, if $r = a + ib$, and a and b are real, then

$$\mathrm{Re}(r) = a, \quad \text{and} \quad \mathrm{Im}(r) = b.$$

As an example, $\mathrm{Re}(5 - 16i) = 5$, and $\mathrm{Im}(5 - 16i) = -16$. Finally, two complex numbers are equal only when their respective real and imaginary parts are equal. So, for example, to state that $e^{i\theta} = \frac{1}{2}\sqrt{2}(1 - i)$ requires that, using Euler's formula, $\cos\theta = \frac{1}{2}\sqrt{2}$ and $\sin\theta = -\frac{1}{2}\sqrt{2}$. ■

Example 2: $e^{i\pi} = \cos\pi + i\sin\pi = -1$.

This shows that the exponential function can be negative. Moreover, since $e^{i\pi} = -1$ then, presumably, $\ln(-1) = i\pi$ (i.e., you can take the logarithm of a negative number). This is true, but there are complications related to the periodicity of the trigonometric functions, and to learn more about this you should take a course in complex variables. ■

Example 3: $e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = i$. ■

Example 4: Assuming θ and φ are real-valued, then

$$\begin{aligned}e^{i\theta}e^{i\varphi} &= (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi) \\ &= \cos\theta\cos\varphi - \sin\theta\sin\varphi + i(\cos\theta\sin\varphi + \sin\theta\cos\varphi) \\ &= \cos(\theta + \varphi) + i\sin(\theta + \varphi) \\ &= e^{i(\theta+\varphi)}. \quad ■\end{aligned}$$

Example 5: Assuming r is real-valued, then

$$\begin{aligned}\frac{d}{dt}e^{irt} &= \frac{d}{dt}(\cos rt + i\sin rt) \\ &= -r\sin rt + ir\cos rt \\ &= ir(\cos rt + i\sin rt) \\ &= ire^{irt}. \quad ■\end{aligned}$$

The next step is to extend Euler's formula to a general complex number. With this in mind, let $z = x + iy$, where x and y are real-valued. Using the usual law of exponents,

$$\begin{aligned} e^z &= e^{x+iy} = e^x e^{iy} \\ &= e^x \big(\cos y + i \sin y \big). \end{aligned} \tag{3.14}$$

The above expression is what we need for solving differential equations.

3.4.2 ▪ Second Representation

We return to the general solution given in (3.12). With (3.14), we get the following

$$\begin{aligned} y &= c_1 e^{(-2+3i)t} + c_2 e^{(-2-3i)t} \\ &= c_1 e^{-2t} \big(\cos 3t + i \sin 3t \big) + c_2 e^{-2t} \big(\cos 3t - i \sin 3t \big) \\ &= (c_1 + c_2) e^{-2t} \cos 3t + i(c_1 - c_2) e^{-2t} \sin 3t. \end{aligned}$$

We have therefore shown that the general solution can be written as

$$y(t) = d_1 e^{-2t} \cos 3t + d_2 e^{-2t} \sin 3t. \tag{3.15}$$

It is not difficult to check that the functions $\bar{y}_1 = e^{-2t} \cos 3t$ and $\bar{y}_2 = e^{-2t} \sin 3t$ are solutions of (3.11), and they have a nonzero Wronskian. Moreover, since $\bar{y}_1$ and $\bar{y}_2$ do not involve complex numbers, then d_1 and d_2 in the above formula are arbitrary real-valued constants.

3.4.3 ▪ Third Representation

There is a third way to write the general solution that can be useful when studying vibration, or oscillation, problems. This comes from making the observation that given the values of d_1 and d_2 in (3.15), we can write them as a point in the plane (d_1, d_2). Using polar coordinates, it is possible to find R and φ so that $d_1 = R \cos \varphi$ and $d_2 = R \sin \varphi$. In this case,

$$\begin{aligned} y &= d_1 e^{-2t} \cos 3t + d_2 e^{-2t} \sin 3t \\ &= R e^{-2t} \big(\cos \varphi \cos 3t + \sin \varphi \sin 3t \big) \\ &= R e^{-2t} \cos(3t - \varphi). \end{aligned} \tag{3.16}$$

This last expression is the formula we are looking for. In this representation of the general solution, R and φ are arbitrary constants with $R \geq 0$. The advantage of this form of the general solution is that it is much easier to sketch the solution, and to determine its basic properties. Its downside is that it can be a bit harder to find R and φ from the initial conditions than the other two representations.

3.5 ▪ Summary for Solving a Homogeneous Equation

To solve

$$y'' + by' + cy = 0, \tag{3.17}$$

where b and c are constants, assume $y = e^{rt}$. This leads to solving the characteristic equation $r^2 + br + c = 0$, and from this the resulting general solution is given below.

Two Real Roots: $r = r_1, r_2$ (with $r_1 \neq r_2$).

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{3.18}$$

One Real Root: $r = \lambda$.

$$y = c_1 e^{\lambda t} + c_2 t e^{\lambda t} \tag{3.19}$$

Complex Roots: $r = \lambda \pm i\mu$ (with $\mu \neq 0$). Any of the following can be used:

$$y = c_1 e^{(\lambda + i\mu)t} + c_2 e^{(\lambda - i\mu)t}, \text{ where } c_1, c_2 \text{ are complex-valued,} \tag{3.20}$$

$$y = d_1 e^{\lambda t}\cos(\mu t) + d_2 e^{\lambda t}\sin(\mu t), \text{ where } d_1, d_2 \text{ are real-valued,} \tag{3.21}$$

$$y = R e^{\lambda t}\cos(\mu t - \varphi), \text{ where } R, \varphi \text{ are constants with } R \geq 0. \tag{3.22}$$

In what follows, (3.21) is used. The exception is in Section 3.10, where (3.22) is preferred because it is easier to sketch.

Example 1: Find a general solution of $y'' + 2y' - 3y = 0$.

Answer: The assumption that $y = e^{rt}$ leads to the characteristic equation $r^2 + 2r - 3 = 0$. The solutions of this are $r = -3$ and $r = 1$. Therefore, a general solution is $y = c_1 e^{-3t} + c_2 e^t$. ■

Example 2: Find the solution of the IVP: $y'' + 2y' = 0$ where $y(0) = 3$ and $y'(0) = -4$.

Answer: The assumption that $y = e^{rt}$ leads to the characteristic equation $r^2 + 2r = 0$. The solutions of this are $r = -2$ and $r = 0$. Therefore, a general solution is $y = c_1 e^{-2t} + c_2$. To satisfy $y(0) = 3$ we need $c_1 + c_2 = 3$, and for $y'(0) = -4$ we need $-2c_1 = -4$. This gives us that $c_1 = 2$, and $c_2 = 1$. Therefore, the solution is $y = 2e^{-2t} + 1$. ■

Example 3: Find the solution of the IVP: $y'' - 2y' + 26y = 0$ where $y(0) = 1$ and $y'(0) = -4$.

Answer: The characteristic equation is $r^2 - 2r + 26 = 0$, and the solutions of this are $r = 1 + 5i$ and $r = 1 - 5i$. Using (3.21), since $\lambda = 1$ and $\mu = 5$, the general solution has the form

$$y = d_1 e^t \cos(5t) + d_2 e^t \sin(5t).$$

To satisfy the initial conditions we need to find y', which for our solution is

$$y' = (d_1 + 5d_2)e^t \cos(5t) + (-5d_1 + d_2)e^t \sin(5t).$$

So, to satisfy $y(0) = 1$ we need $d_1 = 1$, and for $y'(0) = -4$ we need $d_1 + 5d_2 = -4$. This means that $d_2 = -1$, and therefore the solution of the IVP is $y = e^t \cos(5t) - e^t \sin(5t)$. ■

Example 4: Find the solution of the IVP: $y'' - 9y = 0$ where $y(0) = -2$ and $y(t)$ is bounded for $0 \le t < \infty$.

Answer: The assumption that $y = e^{rt}$ leads to the quadratic equation $r^2 = 9$. The solutions of this are $r = -3$ and $r = 3$. Therefore, a general solution is $y = c_1 e^{-3t} + c_2 e^{3t}$. To satisfy $y(0) = 1$ we need $c_1 + c_2 = -2$. As for boundedness, e^{-3t} is a bounded function $0 \le t < \infty$ but e^{3t} is not. This means we must take $c_2 = 0$. The resulting solution is $y = -2e^{-3t}$. ■

As you might have noticed, in the above examples the formula for the roots in (3.10) was not used. The reason is that it is much easier to remember the way the characteristic equation is derived (by assuming $y = e^{rt}$, etc) than by trying to remember the exact formula for the roots.

Exercises

1. Assuming that $z_1 = 1 + i$, and $z_2 = e^{2+i\frac{\pi}{6}}$, find $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$:

   a) $z = z_1 - 8$
   b) $z = 2iz_1$
   c) $z = z_2$
   d) $z = z_1 + 4z_2$
   e) $z = z_1 z_2$
   f) $z = (z_2)^6$

2. Assuming θ and φ are real-valued, show that the following hold:

   a) $\dfrac{1}{i} = -i$
   b) $\dfrac{1}{a + ib} = \dfrac{a - ib}{a^2 + b^2}$
   c) $e^{i\theta} \neq 0, \forall\theta$
   d) $e^{-i\theta} = \dfrac{1}{e^{i\theta}}$
   e) $(e^{i\theta})^2 = e^{2i\theta}$
   f) $e^{i(\theta+2\pi)} = e^{i\theta}$
   g) $e^{i(\theta-\varphi)} = \dfrac{e^{i\theta}}{e^{i\varphi}}$
   h) $\int e^{i\theta} d\theta = -ie^{i\theta} + c$
   i) $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$
   j) $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$